On Dirichlet's L-functions

of

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ABSTRACT

A survey of the developments on the Dirichlet's L-functions and Riemann's zeta-function in the recent decade is given.

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0. Introduction

In 1837, Dirichlet [1] proved that, if a and b are two relative prime positive integers, the polynomial $ax + b$ gives infinitely many primes as x runs through all the positive integers. This result is now known as Dirichlet's theorem. To prove this theorem, Dirichlet went outside the realm of integers introduced the L-function $L(s, \chi)$ and used tools of analysis. By so doing he laid the foundations for a new branch of mathematics called analytic number theory. $L(s, \chi)$ and $\zeta(s)$ are two important functions in this subject, especially useful in the study of the distribution of the primes.

In the study of L-functions one is often confronted with the following problems:-

(1) analytic continuation and functional relations,

(2) asymptotic formulas, mean modulus, modulus, and the distribution of the values of $L(s, \chi)$, namely,

(3) the location of the zeros of $L(s, \chi)$, in particular

(a) nonvanishing results and (b) zero-density results,

(4) the determination of the values of $L(s, \chi)$ at special points.

W.J. LeVeque [1] collected together these reviews in number theory through appeared in the Mathematical Reviews between 1940 and 1972. The results on $L(s, \chi)$ and $\zeta(s)$ which appeared in this period are included in this collection which is one of the important starting point of this thesis.

The aim of this thesis is to give a survey on the results on $L(s, \chi)$ and $\zeta(s)$ obtained in the last decade, in particular those reviewed in the

Mathematical Review from January 1972 to December 1981. We hope this would serve as a useful supplement to the above mentioned work of LeVegue.

The arrangement in this thesis is slightly different from LeVeque [1]. In Chapter 1, we state some basic properties which will be used throughout the thesis. In Chapter 2, we give the functional properties of $L(s, \chi)$ and $\zeta(s)$ (cf. LeVeque [1] M05 M40).

Those special properties including mean moduli , order of magnitude, asymptotic behaviour are collected in Chapter 3 (cf. LeVeque [1] M10, M15, M30, M40). Chapter 4 contains the results on the bounds of $L(1, \chi)$ and the nonvanishing regions (cf. LeVeque [1] M20, M25). In Chapter 5, the results of zeros of $L(s, \chi)$ and $\zeta(s)$ are given, in particular, the zero density problem is discussed in §5.3 and §5.5. (cf. LeVeque [1] M20, M25).

The special values at integers are given in Chapter 6 (cf. Leveque [1] M10), and in the last chapter, we consider some related zeta functions such as Hurwitz zeta function, Lerch zeta function, secondary zeta function and Epstein zeta function. (cf. LeVeque M35, M40).

It is clear from this survey that despite of a lot of efforts by many mathematicians, the major conjectures and problems on $L(s, \chi)$ have not been solved. Improved techniques and more careful calculations lead to improved constants. But definite solutions remained to be obtained.

Notation.

1. $e(z) = e^{2\pi iz}$.

2. $\sum_{\chi \bmod q}$: sum for all characters with modulus q .

3. $\sum^{*}_{\chi \bmod q}$: sum for all primitive characters with modulus q .

4. Mangoldt function $\Lambda(u) = \begin{cases} \log p & n = p^k, \ p \text{ prime} \\ 0 & \text{otherwise.} \end{cases}$

5. ϕ : Euler function.

6. $\pi(x) = \sum_{p \le x} 1$.

7. $f \ll g \Leftrightarrow$ there exist an independent contant A s.t. $|f| \le Ag$.

8. $f \ll_\varepsilon g \Leftrightarrow$ there exists an constant A which is depended on ε s.t. $|f| \le Ag$.

9. $f - g \ll h \Leftrightarrow f = g + O(h)$.

10. $f = o(g) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.

11. $f(x) = \Omega(x)$: $|f(x)| > cx$ for arbitrarity large values of x .

$f(x) = \Omega_+(x)$: $f(x) > cx$,,

$f(x) = \Omega_-(x)$: $f(x) < -cx$,,

12. II §2.2 = paragraph 2.2 of Chapter 2.

13. II(2.2) = equation (2.2) of Chapter 2.

CONTENTS

Chapter I. BASIC PROPERTIES

For the convenience of reference we first list the well known properties of Dirichlet L-functions.

§1.1. Definition and functional equation.

Definition. We say that $\chi(n)$ is a Dirichlet's character (or character) to the modulus q if it is a complex-valued function of integers satisfying the conditions:

(i) $\chi(1) \neq 0$

(ii) $\chi(n) = 0$ if $(n, q) > 1$

(iii) if $n \equiv m \pmod q$, then $\chi(n) = \chi(m)$

(iv) $\chi(nm) = \chi(n)\chi(m)$

If we set $\chi(n) = 1$ for $(n, q) = 1$, it is obvious that $\chi(n)$ is a Dirichlet's character to the modulus q and is called the principal character, denoted by χ_0.

It is clear, from the definition, that any Dirichlet's character to the modulus q is a periodic function of n with period q. It is possible, however, that for values of n restricted by the condition $(n, q) = 1$, the function $\chi(n)$ may have a period less than q. If so, we say that χ is imprimitive, and otherwise primitive.

Throughout this paper, except mentioned to the contrary, let $\chi(n)$ be a character of modulus q.

1.1.1 χ Dirichlet character mod $q \geq 1$, $s = \sigma + it$, the series

$$L(s, \chi) = \sum_n \frac{\chi(n)}{n^s} \tag{1.1}$$

is convergent for $\sigma > 1$.

1.1.2 For $\sigma > 1$, χ Dirichlet character mod q

$$L(s, \chi) = \prod_{p \nmid q} (1 - \frac{\chi(p)}{p^s})^{-1} = \prod_p (1 - \frac{\chi(p)}{p^s})^{-1} \tag{1.2}$$

(Prachar [1] p.103, Davenport [1] p.32).

1.1.3 For $\chi \neq \chi_o$, $\sum_n \frac{\chi(n)}{n^s}$ convergent in $0 < \sigma \leq 1$. $L(s, \chi_o)$ has a pole at $s = 1$ and has expansion

$$L(s, \chi_o) = \frac{a_{-1}}{s - 1} + a_o + a_1(s - 1) + \ldots \tag{1.3}$$

where $a_{-1} = \prod_{p \nmid q} (1 - \frac{1}{p}) = \frac{\psi(q)}{q}$

$L(s, \chi)$ can be analytically continued into $0 < \sigma \leq 1$ except $L(s, \chi_o)$ has a pole of order 1 at $s = 1$. (Prachar [1] p.105).

1.1.4 If χ is an imprimitive character mod q induced by the primitive character χ_1 mod q_1 i.e. $q_1 | q$ and

$$\chi(n) = \begin{cases} \chi_1(n) & \text{if } (n, q) = 1 \\ 0 & \text{if } (n, q) > 1 \end{cases} ,$$

then $$L(s, \chi) = L(s, \chi_1) \prod_{p|q} (1 - \frac{\chi_1(p)}{p^{-s}}) \tag{1.4}$$

(Prachar [1] p.127, Davenport [1] p.39).

1.1.5 Let χ be a primitive character mod q

Put $a = a_\chi = \frac{1}{2}(1 - \chi(-1))$

$$G(\chi) = \sum_{m=1}^{q} \chi(m) e\left(\frac{m}{q}\right)$$

$$\varepsilon_\chi = \frac{i^a q^{\frac{1}{2}}}{\tau_\chi}$$

$$\xi(s, \chi) = \left(\frac{q}{\pi}\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi) .$$

Then

$$\xi(1 - s, \bar{\chi}) = \varepsilon_\chi \, \xi(s, \chi) \ , \ |\varepsilon_\chi| = 1 \tag{1.5}$$

and $L(s, \chi)$ analytic in the entire plane except when $\chi = \chi_o$ and $s = 1$ and for $\sigma < 0$, the only zeros of $L(s, \chi)$ are

$$S = \begin{cases} 0 \ , -2, -4, -6, \ldots \\ -1, -3, -5, -7, \ldots \end{cases} \qquad \text{if} \quad a_\chi = \begin{cases} 0 \\ 1 \end{cases}$$

(Prachar [1] p.207, Davenport [1] p.71-73).

1.1.6 (Approximate functional equation) Let $\gamma > 0$, $t \geq 2\pi\gamma^2$, $\sigma \leq \alpha \leq t(2\pi\gamma)^{-1}$, $2\pi\alpha\beta = t$, $0 \leq \alpha \leq 1$ and $s = \sigma + it$. The character χ is primitive mod q, where q is the least modulus of χ (if $q = 1$, then $\chi \equiv 1$). Then

$$L(s, \chi) = \sum_{1 \leq m \leq \alpha q} \frac{\chi(m)}{m^s} + H(s, \chi) \sum_{1 \leq m \leq \beta} \frac{\bar{\chi}(m)}{m^{1-s}} + R(s, \chi) \tag{1.6}$$

where

$$R(s, \chi) \ll \left(1 + \frac{1}{\gamma}\right)^2 \left(1 + \frac{1}{\gamma^2}\right)^2 k^{1-\sigma} \left(\alpha^{-\sigma} + \alpha^{1-\sigma} t^{-\frac{1}{2}} + \alpha^{-1-\sigma} t^{\frac{1}{2}}\right)$$

$$H(s, \chi) = iT(\chi)\chi(-1) k^{\frac{1}{2}-s} (2\pi)^{s-1} e^{-\frac{1}{2}\pi s i} \Gamma(1 - s)$$

$$T(\chi) = k^{-\frac{1}{2}} \sum_{1 \leq a \leq k} \chi(a) \exp 2\pi i \frac{a}{k} \ .$$

The second term in the right hand side is to be omitted if $\beta < 1$.

(cf: Tchudahoff [1] Chandrasekharan & Narasimhan [1], Tatuzawa [1], Davies [1], Lavrik [1]).

1.1.7 Let $\chi\ (\neq \chi_0)$ be a primitive character mod q

Then

$$\xi(s, \chi) = e^{A+Bs} \Pi(1 - \frac{s}{\rho})e^{\frac{s}{\rho}} \qquad (1.7)$$

where ρ runs over all the zeros of $L(s, \chi)$ in the critical strip $0 \le \sigma \le 1$, $e^A = \xi(0, \chi)$ and $B = \frac{\xi'(0, \chi)}{\xi(0, \chi)}$.

and

$$\frac{L'(s, \chi)}{L(s, \chi)} = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'(\frac{s+a}{a})}{\Gamma(\frac{s+a}{2})} + B + \sum_\rho(\frac{1}{s-\rho} + \frac{1}{\rho}) \qquad (1.8)$$

(Davanport [1] 85 Prachar [1] 218)

1.1.8 χ arbitrary character

$$\frac{L'(s, \chi)}{L(s, \chi)} = \Sigma\frac{1}{s-\rho_0} + \frac{E_0}{s-1} + \frac{\nu_0}{s} + \frac{a}{s+1} \qquad (1.9)$$

$$+ O(\log q(|t| + 2)) \ , \quad -1 \le \sigma \le 2 \quad \text{(Prachar [1] 225)}$$

where the sum runs over all zeros $\rho_0 = \beta_0 + i\gamma_0$ of $L(s, \chi)$ in $0 \le \sigma < 1$ such that $\rho_0 \neq 0$ and $|\gamma_0 - t| \le 1$.

$$\frac{L'(s, \chi)}{L(s, \chi)} << \log q(|s| + 2) \quad \text{for} \quad \sigma \le -\frac{1}{2}. \quad \text{(Prachar [1] 227)} \qquad (1.10)$$

1.1.9

$$\frac{1}{L(s, \chi)} = \sum_n \mu(n)\frac{\chi(n)}{n^s}$$

$$= O(\frac{1}{\sigma - 1}) \quad . \qquad (1.11)$$

(see Prachar [1] p.113, p.116).

§1.2. Mean moduli

We are interested in results of the form

$$\sum_{\chi} \int_{-T}^{T} |L(\tfrac{1}{2} + it, \chi)|^{2k} dt << \psi(q)T(\log_q T)^{c(k)}$$

where k is a positive integer and $c(k)$ a constant depending on k .

1.2.1 Let $q > 2$, $s = \frac{1}{2} + it$. Then

$$\sum_{\chi \neq \chi_0} |L(s, \chi)|^2 << \psi(q)|s| \log^2 q|s| \tag{1.12}$$

$$\sum_{\chi \neq \chi_0} |L'(s, \chi)|^2 << \psi(q)|s| \log^4 q|s| \tag{1.13}$$

(Pan [1] III§4)

1.2.2 If $T \geq 2$ and $|\sigma - \frac{1}{2}| \leq \frac{1}{\log_q T}$. Then

$$\sum_{\chi} \int_{-T}^{T} |L(\tfrac{1}{2} + it, \chi)|^4 dt << \psi(q)T \log_q^4 T \tag{1.14}$$

(Montgomery [2] Th.10.1)

(Pan [1] III§3)

§1.3. Nonvanishing

1.3.1 For all χ , if $\sigma > 1$ then $L(s, \chi) \neq 0$ (Prachar [1] p.104)

1.3.2 $L(1 + it, \chi) \neq 0$ and

$L(1, \chi) > 0$ for $\chi \neq \chi_0$ (Prachar [1], pp.107, 108)

1.3.3 (Siegel [1]) (1) For any $\varepsilon > 0$ there exists a positive number $C_1(\varepsilon)$ such that if χ is a real primitive character mod q , then

$$L(1, \chi) > C_1(\varepsilon)q^{-\varepsilon} \tag{1.15}$$

(Prachar [1] p.146)

(Davenport 1 p.130)

(2) For any $\varepsilon > 0$ there exists a positive number $C_3(\varepsilon)$ such that if χ is a real non-principal character mod q , then

$L(s, \chi) \neq 0$ for $\sigma > 1 - C_3(\varepsilon)q^{-\varepsilon}$ (Prachar [1] p.144)

(Davenport [1] p.130)

1.3.4 χ real primitive mod $q \geq 3$.

$L(s, \chi) \neq 0$ for $s > 1 - cq^{-\frac{1}{2}}(\log\log q)^{-1}$ if $\chi(-1) = -1$

$1 - cq^{-\frac{1}{2}}(\log\log q)^{-1}\log q$ if $\chi(-1) = 1$

(Devenport, Nachr, Göttingon Kl.II 1966, 203-212, MR36#120).

1.3.5 $L(s, \chi) \neq 0$ for $\chi^2 \neq \chi_0$, $\sigma \geq 1 - \frac{c_1}{\log 2q}$, $|t| \leq 5$ (Prachar [1] p.118, p.120 p.122)

$\chi^2 = \chi_0$, $\sigma \geq 1 - \frac{c_2}{\log 2q}$, $0 < |t| \leq 5$,

s not real and atmost one simple

zero for, $\sigma \geq 1 - \frac{c_2}{\log 2q}$, $t = 0$.

1.3.6 There exists a positive absolute constant c with the following property (1) If χ is a complex character mod q then $L(s, \chi)$ has no zero in the region defined by

$$\sigma \geq \begin{cases} 1 - \dfrac{c}{\log q|t|} & \text{if } |t| \geq 1 \\ 1 - \dfrac{c}{\log q} & \text{if } |t| \leq 1 \end{cases}$$

(2) If χ is a real non-principal character, the only possible zero of $L(s, \chi)$ in this region is a single (simple) real zero.

(3) For atmost one of the real non-principal characters χ mod q can $L(s, \chi)$ have a zero in this region. (Davenport [1] p.16, 98)

Prachar, [1] p.117

1.3.7 If $q \geq 1$, $s = \sigma + it$; for at most one of the characters χ mod q can $L(s, \chi)$ have a zero in the region

$$\sigma \geq 1 - \frac{c}{\log q(|t| + 2)} .$$

If such an exceptional character, say $\bar{\chi}$ exists, then $\bar{\chi}$ must be a real nonprincipal character and the only possible zero of $L(s, \bar{\chi})$ in this region is a single simple real zero, say $\tilde{\beta}$, called the Siegel zero. (Prachar [1] p.130, Deuring-Heilboronn phenomenon). Moreover if an exceptional zero $\tilde{\beta}$ exists, then the function $\prod_{\chi} L(s, \chi)$ has no zero other than $\tilde{\beta}$ in the region.

$$\sigma \geq 1 - \frac{c_1}{\log q(|t| + 2)} \log \frac{c_2 e}{\tilde{\delta} \log q(|t| + 2)} ,$$

$$\tilde{\delta} \log q(|t| + 2) \leq c_2 \quad \text{where} \quad \tilde{\delta} = 1 - \tilde{\beta}$$

(Pan [1] §3. Th.7).

1.3.8 If $Q > 1$, $s = \sigma + it$, for atmost one of the characters χ mod q , $q \leq Q$ can $L(s, \chi)$ have a zero in the region

$$\sigma \geq 1 - \frac{c}{\log Q(|t| + 2)} .$$

If such an exceptional character, say $\tilde{\chi}$ exists, then $\tilde{\chi}$ must be a real primitive character mod $\tilde{q} \leq Q$ and the only possible zero of $L(s, \tilde{\chi})$ in this region is a single simple real zero, say $\tilde{\beta}$.

Moreover if such an exceptional zero $\tilde{\beta}$ exists, then the function $\prod_{q \leq Q} \prod^*_{\chi} L(s, \chi)$ has no zero other than $\tilde{\beta}$ in the region

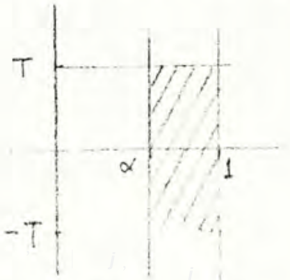
$$\sigma \geq 1 - \frac{c_1}{\log Q(|t| + 2)} \log \frac{c_2}{\tilde{\delta} \log Q(|t| + 2)} ,$$

$$\tilde{\delta} \log Q(|t| + 2) \leq c_2 , \quad \tilde{\delta} = 1 - \tilde{\beta} . \qquad \text{(Pan [1] §3, Th.2)}$$

§1.4 Density of zeros

Let $N(\alpha, T, \chi)$ be the number of zeros $\sigma + it$ of the function

$L(s, \chi)$ in the rectangle $R(\alpha, T)$: $\alpha \le \sigma \le 1$, $-T \le t \le T$. If χ is principal then we write $N(\alpha, T, \chi) = N(\alpha, T)$ for the number of zeros of the zeta function in this rectangle. Let $N(T, \chi)$ denote the number of zeros of $L(s, \chi)$ in the rectangle $0 < \sigma < 1$, $|t| < T$. Various bounds have been given for $N(T, \chi)$, $N(\alpha, T, \chi)$, $\sum_\chi N(\alpha, T, \chi)$ and $\sum_{q \le Q} \sum_\chi^* N(\alpha, T, \chi)$. (here: $\sum_\chi$ denotes a sum over all characters modulo q and $\sum_\chi^*$ denotes a sum over all primitive characters modulo q) .



1.4.1 For $\alpha > \frac{1}{2}$ we have $N(\alpha, T) \ll_\alpha T$ (Bohr and Landau [1], see also Titchmursh [1] Th.9.15(A)) and $N(\frac{1}{2}, T) \gg T \log T$ (Titchmursh [1]§10.7, 10.22). This established that almost all zeros of $\zeta(s)$ are near the line $\alpha = \frac{1}{2}$.

1.4.2 For $T \ge 2$, we have

$$\frac{1}{2}N(T, \chi) = \frac{T}{2\pi} \log \frac{qT}{2\pi} - \frac{T}{2\pi} + O\,(\log T + \log q) \tag{1.16}$$

(Davenport [1] §16)

1.4.3 Let $\overline{N}(\alpha, T, \chi)$ be the number of zeros $\sigma + it$ of $L(s, \chi)$ with $\sigma \ge \alpha$ and $|t| \le T$. Then for $T \ge 1$

$$\overline{N}(\alpha, T, \chi) \ll q^{2\alpha-1} T^{\frac{4\alpha}{2\alpha-1}} (\log T)^{7.5} + q^2(\log q + 1) \tag{1.17}$$

[Tschudakoff [1]: Theorem 2]

1.4.4 Suppose that $q \ge 1$ and $T \ge 2$. For $\frac{1}{2} \le \alpha \le \frac{4}{5}$ we have

$$\sum_\chi N(\alpha, T, \chi) \ll (qT)^{\frac{3}{2-\alpha}(1-\alpha)} (\log qT)^9 \tag{1.18}$$

and for $\frac{4}{5} \le \alpha \le 1$ we have

$$\sum_{\chi} N(\alpha, T, \chi) << (qT)^{\frac{2}{\alpha}(1-\sigma)} (\log qT)^{14} \tag{1.19}$$

(Montgomery [2], Th.12.1)

1.4.5 If $\frac{4}{5} \le \alpha \le 1$, $T \ge 1$ then

$$\sum_{\chi} N(\alpha, T, \chi) <<_{\varepsilon} (qT)^{(2+\varepsilon)(1-\alpha)} \quad \text{(Jutila [7])} \tag{1.20}$$

1.4.6 Suppose that $Q \ge 1$ and $T \ge 2$. For $\frac{1}{2} \le \alpha \le \frac{4}{5}$ we have

$$\sum_{q \le Q} \sum_{\chi}{}^{*} N(\alpha, T, \chi) << (Q^2 T)^{\frac{3}{2-\alpha}(1-\alpha)} (\log QT)^{9} \tag{1.21}$$

and for $\frac{4}{5} \le \alpha \le 1$ we have

$$\sum_{q \le Q} \sum_{\chi}{}^{*} N(\alpha, T, \chi) << (Q^2 T)^{\frac{2}{\sigma}(1-\alpha)} (\log QT)^{14} \tag{1.22}$$

(Montgomery [2], Th.12.2)

1.4.7 If $\frac{4}{5} \le \alpha \le 1$, $T \ge 1$ then

$$\sum_{q \le Q} \sum_{\chi}{}^{*} N(\alpha, T, \chi) <<_{\varepsilon} (Q^2 T)^{(2+\varepsilon)(1-\alpha)} \quad \text{(Jutila [7])} \tag{1.23}$$

1.4.8 If $\frac{1}{2} \le \alpha \le 1$, $T \ge 2$ then

$$\sum_{q \le T} \sum_{\chi}{}^{*} N(\alpha, T, \chi) \le c_1 T^{c_2(1-\alpha)} \quad \text{(Pan [1], §4 Th.3)} \tag{1.24}$$

II. FUNCTIONAL PROPERTIES.

§2.0. The functional equation relating $L(s, \chi)$ and $L(1-s, \bar{\chi})$ was first given by Hurwitz in 1882 (Werke I pp.72-88) for the case of real characters. The standard proof of the equation given in §1.5 follows the method used by de la Vallée Poussin in 1896, which is an extension of that Riemann used in the proof of the functional equation of ζ-function by first multiplying by the Γ-function.

Using the functional equation (§1.5) one gets Dirichlet series expansion of $L(s, \chi)$ for $\text{Re } s > 1$ and $\text{Re } s \leq 0$. Similar expansion in the critical strip $0 < \text{Re } s < 1$ is called the approximate functional equation. Hardy and Littlewood first obtained approximate functional equation for $\zeta(s)$ in 1929 and Suetuna [1] gave the approximate functional for Dirichlet L-function in 1932. (see §1.6).

Before we proceed to results after 1972, we must state three other interesting theorems. One is due to TURÁN [2] : if in a certain half-plane a function $f(s)$ can be represented both by a convergent Dirichlet series $\Sigma a_n \chi(n) n^{-s}$ with monotonic coefficients $a_n > 0$ and by an absolutely convergent Euler-product $\Pi(1 - \varepsilon_p \chi(p) p^{-s})^{-1}$ with $\varepsilon_p > 0$, then there exist a real number c such that $f(s) = L(s + c, \chi)$. The second is due to Apostol [2] : if $L(s, \chi)$ satisfies the usual functional equation then χ is primitive. The third, due to Bochner [1] deals with the maximum number of linearly independent solutions (ψ, Ψ) of the functional equation

$$\pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \psi(s) = \pi^{-\frac{(\delta-s)}{2}} \Gamma(\tfrac{\delta-s}{2}) \Psi(\delta - s)$$

where $\psi(s) = \Sigma a_n \lambda_n^{-s}$ and $\Psi(s) = \Sigma b_n \mu_n^{-s}$.

§2.1. We first consider those results concerning L-functions.

By using contour integration, Berndt 3 give a simple new proof of the functional equations (1.5) of the Dirichlet L-function $L(s, \chi)$ for a nonprincipal primitive character χ modulo q. He started with a function

$$F(z) = \frac{\pi e^{-\pi iz} G(z, \overline{\chi})}{G(\overline{\chi}) s \sin(\pi z)} \qquad (s > 1)$$

where

$$G(z, \chi) = \sum_{j=1}^{q-1} \chi(j) e\left(\frac{jz}{q}\right) \qquad \text{denote a Gaussian sum, and}$$

$G(\chi) = G(1, \chi)$.

If m is a positive integer, let C_m denote the positively oriented, closed contour consisting of Γ_m, the right half of the circle with centre $(0, 0)$ and radius $m + \frac{1}{2}$, together with the vertical diametor indented at the origin by a semi circle Γ_ε of radius $\varepsilon < 1$ in the right half plane. The author then considered the contour integral and by the residue theorem

$$\frac{1}{2\pi i} \int_{C_m} F(z)dz = \sum_{n=1}^{m} \chi(n) n^{-s} \tag{2.1}$$

by sending m to infinity, the L.H.S. of (2.2) becomes the Dirichlet L-function, and after calculating the integral of (2.2), by using the formula

$$\Gamma(s)L(s, \chi) = \int_0^\infty \frac{x^{s-1} G(iqx/2\pi, \chi)dx}{1 - e^{-qx}}$$

of Ayoub [1], the functional equation in the following form $(\sigma > 0)$

$$L(1-s, \chi) = (k/2\pi)^s k^{-1} G(\chi)\Gamma(s)L(s, \overline{\chi})\{e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\} .$$

is established.

§2.2. If $\chi_1, \ldots, \chi_k$ are arbitrary Dirichlet characters, and g is a function satisfying a condition of the form $g(t) = O(t^*)$, M.I. Pulatova [1] showed that for $s_1, \ldots, s_k$ in appropriate half-planes and $F(s_1, \ldots, s_k) = \sum_{n_1, \ldots, n_k=1}^{\infty} g((n_1, \ldots, n_k))\chi_1(n_1)n_1^{-s_1} \ldots \chi_k(n_k)n_k^{-s_k}$ we have $L^{-1}(s_1, \chi_1) \ldots L^{-1}(s_k, \chi_k)F(s_1, \ldots, s_k) = \sum_{n=1}^{\infty} c_n n^{-s_1 - \ldots - s_k}$, where $c_n = \chi_1(n) \ldots \chi_k(n)\sum_{d/n}\mu(n/d)g(d)$.

§2.3. Let $L_1(s), \ldots, L_n(s)$ be Dirichlet L-functions corresponding to pairwise nonequivalent characters. In S.M. Voronin [4], the author studied the map $\gamma : R^2 \to C^n$ given by $\gamma(\sigma, T) = [L_1(\sigma+iT), \ldots, L_n(\sigma+iT)]$ and showed that for fixed $\sigma_o \in (\frac{1}{2}, 1]$, the image of any set of the form $\{<\sigma_o, T> : 0 < \alpha_j < \{T \log P_j/2\pi\} < \beta_j < 1\}$ (where the α_j, β_j are given real numbers and the P_j are given primes) is dense in C^n. This result allows him to show that if $F_o, \ldots, F_m$ are continuous and not all of them vanish identically, then $\sum_{k=0}^{m} s^k F_k(L_1(s), \ldots, L_n(s))$ cannot vanish identically. S.M. Voronin [5] gave a summary for S.M. Voronin [4].

§2.4. F. Oberhettinger; K.L. Soni [1] centers around three classical summation formulas due to Poisson, Sierpinski and Voronoi respectively.

$$\frac{1}{2}f(0) + \sum_1^{\infty} f(n) = G(0) + \sum_1^{\infty} G(n) \qquad (2.3)$$

$$\sum_o^{\infty} r(n)f(n) = \sum_o^{\infty} r(n)G(n) \qquad (2.4)$$

$$-\frac{1}{4}f(0) + \sum_1^{\infty} d(n)f(n) = \int_o^{\infty} (2\gamma + \log x)f(x)dx + \sum_1^{\infty} d(n)G(n) \qquad (2.5)$$

where
$$G(y) = \int_o^{\infty} f(x)K(xy)dx \quad . \qquad (2.6)$$

The kernel function in the transform (2.6) of the function $f(x)$ occuring in the sums (2.3) to (2.5) is given by

$$K(u) = 2\cos(2\pi u) \qquad \text{in } (2.3) \tag{2.7}$$

$$K(u) = \pi J_o(2\pi\sqrt{u}) \qquad \text{in } (2.4)$$

$$K(u) = 4K_o(4\pi\sqrt{u}) - 2\pi Y_o(4\pi\sqrt{u}) \qquad \text{in } (2.5)$$

In (2.7), J_o, Y_o, K_o are the Bessel function, the Neumann function and the modified Hankel function all of order zero. In (2.4), $r(n)$ is the number of integer solutions of $p^2 + q^2 = n$. In (2.5), $d(n)$ is the number of divisors of n. Finally, $\gamma = 0.57712\cdots$ is Euler's constant. The functions K in (2.7) are respectively the Fourier cosine transform, the Hankel transform and the divisor transform kernel.

Three classical special cases of (2.3), (2.4), (2.5) are (valid for $\mathrm{Re}\ \tau > 0$)

$$\tau^{\frac{1}{2}} \sum_{-\infty}^{\infty} e^{-n^2\pi\tau} = \tau^{-\frac{1}{4}} \sum_{-\infty}^{\infty} e^{-n^2\pi/\tau}, \tag{2.8}$$

$$\frac{1}{2}\sum_{0}^{\infty} r(n)e^{-n\pi\tau} = \tau^{-\frac{1}{2}} \sum_{0}^{\infty} r(n)e^{-n\pi/\tau}, \tag{2.9}$$

$$\tau^{\frac{1}{2}}[\gamma - \log(4\pi/_\tau)] + 4\sum_{1}^{\infty} d(n)K_o(2n\pi\tau)$$
$$= \tau^{-\frac{1}{2}}[\gamma - \log(4\pi\tau)] + 4\sum_{1}^{\infty} d(n)K_o(2n\pi/\tau), \tag{2.10}$$

which can be obtained by putting $f(x)$ equal to $e^{-\pi\tau x^2}$, $e^{-\pi\tau x}$ and $K_o(2\pi\tau x)$ in (2.3), (2.4), (2.5) respectively.

It has been shown by Hamburger that (2.8) is not only a special case of (2.3) but vice versa (2.3) is also a consequence of (2.8). Moreover, it is shown that besides (2.8) the identities

$$1 + \sum_{1}^{\infty} e^{-2nz\pi} = (\pi z)^{-1} + 2z^{-1}\sum_{1}^{\infty}(z^2+n^2)^{-1}, \tag{2.11}$$

$$\sum_{1}^{M}(y-n) = \frac{1}{2}y(y-1) - \frac{1}{2}\pi^{-2}\sum_{1}^{\infty} n^{-2}(\cos 2\pi ny - 1), \quad m < y < m+1, \tag{2.12}$$

$$2\pi \sum_{-\infty}^{\infty} f(2n\pi) = \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)e^{inu}\,du \qquad (2.13)$$

also hold good and that (2.8) and (2.11) - (2.13) are equivalent to each other. Moreover, all of them are equivalent to the functional equation of the Riemann zeta function under certain assumptions. We note that (2.13) is idential with (2.3) when $f(u)$ is an even function of u. This suggests the possiblity that (2.9) and (2.10) may likewise not just be examples of (2.4) and (2.5) but similar "equivalent" to certain functional equations so that (2.9) and (2.10) imply also (2.4) and (2.5).

In Part I of the paper, the authors generalised (2.9) and (2.10), under certain conditions, such that these new generalised formulas are "equivalent" to the respective functional equations upon specialising parameters:

$$\pi^{-s}\Gamma(s)f(s) = \pi^{s-1}\Gamma(1-s)f(1-s) \qquad (2.14)$$

$$\pi^{-s}[\Gamma(\tfrac{1}{2}s)]^2 f(s) = \pi^{s-1}[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)]^2 f(1-s) \qquad (2.15)$$

the former being the functional equation for

$$f(s) = L(s)\zeta(s) = \frac{1}{4}\sum_{1}^{\infty} r(n)n^{-s} \qquad \mathrm{Re}\ s > 1 \qquad (2.16)$$

the latter for

$$f(s) = \zeta^2(s) = \sum_{1}^{\infty} d(n)n^{-s} \qquad \mathrm{Re}\ s > 1 \qquad (2.17)$$

In Part II the authors gave a similar generalisation of the sum formulas (2.4) and (2.5) and showed that these generalisations are "equivalent" to the same functional equations mentioned above. Speciali- sation of parameters will then show that (2.4) and (2.5) are "equivalent" to (2.14) and (2.15), that is, (2.9) and (2.10) also imply (2.4) and (2.5) respectively.

§2.5. D. Suryanarayana [1] proved the equality

$$\frac{1}{3}\zeta(2) - \frac{1}{4}\zeta(3) + \frac{1}{5}\zeta(4) - \dots = 1 + \gamma/2 - \log\sqrt{2\pi}$$

where γ is Euler's constant.

§2.6. In R.C.R.P. Sita; R.S.A. Siva [1], the authors discussed six identities involving the Riemann zeta function. They first reduce the proofs of these identities to proving just two of them, namely,

$$2\sum_{r=1}^{\infty} r^{-a}\sum_{k=1}^{r} k^{-1} = (a+2)\zeta(a+1) - \sum_{i=1}^{a-2}\zeta(a-i)\zeta(i+1)$$

and
$$\sum_{r=1}^{\infty}\sum_{k=1}^{r} (kr(k+r))^{-1} = (5/4)\zeta(3) .$$

§2.7. Let χ denote a primitive Dirichlet character to the modulus $q > 1$. C. Ryavec [2] gave a representation of $\zeta(s)$:

$$\zeta(s) = -\frac{2^{s-1}}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^2}(\chi(n) + \bar{\chi}(n))\log^{s-1} n - 2^s\sum_{\rho}\frac{\cos\{s \operatorname{Tan}^{-1}(\frac{\gamma}{\beta+1})\}}{|1+\rho|^s} \qquad (2.18)$$

for $\sigma > 1$, where the second sum is over the zeros, $\rho = \beta + i\gamma$, $0 < \beta < 1$, of $L(s, \chi)$. The proof is based on a lemma of A. Weil: Let $\hat{\psi}(w) = \hat{\psi}(u + iv)$ denote any function analytic on $|v| \le 1/_{4\pi} + \delta$ such that $\hat{\psi}(w) = \hat{\psi}(-w)$ and $|\hat{\psi}(w)| \le C(1 + |w|)^{-2-\delta}$ for $C, \delta > 0$. Then if χ is any primitive character to the modulus $q > 1$, with $\chi(-1) = 1$, we have

$$\sum_{\rho}\hat{\psi}\left(\frac{\rho - \frac{1}{2}}{2\pi i}\right) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\frac{1}{2}}}(\chi(n) + \bar{\chi}(n))\psi(\log n) - \psi(0)\log\left(\frac{\pi}{q}\right)$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\psi}\left(\frac{t}{2\pi}\right)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2}\right)dt , \qquad (2.19)$$

where the sum over ρ is over all zeros of $L(s, \chi)$ satisfying $\rho = \beta + i r$,

$0 < \beta < 1$, and where

$$\hat{\psi}(t) = \int_{-\infty}^{\infty} e^{2\pi i t x}\psi(x)dx .$$

The author modified this lemma to give:

Let ψ satisfy the hypothesis of A. Weil's lemma, then

$$\sum_{\rho} \hat{\psi}\left(\frac{\rho - \frac{1}{2}}{2\pi i}\right) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\frac{1}{2}}}(\chi(n) + \overline{\chi}(n))\psi(\log n) - B(\chi)\psi(0)$$

$$+ \int_{0}^{\infty} (\psi(0) - \psi(n))\frac{e^{u/2}}{\sin hu}du \quad , \qquad (2.20)$$

where $B(\chi) = \log(\frac{\pi}{q}) - \frac{\tau'}{\tau}(\frac{1}{4}) = 2\frac{L'}{L}(\frac{1}{2}, \chi)$.

By choosing $\psi(u) = |u|^{s-1}e^{-3|u|/2}$, (2.20) gives (2.18), it is seen that (2.18) is just one example of a wide class of representations of $\zeta(s)$ which result from a choice of $\psi(u)$ of the form $\psi(u) = |u|^{s-1}e^{-|u|/2}\sin h|u|\sum_{n=1}^{\infty} f(n|u|)$, with f suitably restricted; for example, (2.18) results from the simple choice $f(u) = 2e^{-2u}$, $u \geq 0$. In order to obtain a representation of $\zeta(s)$, valid to the left of $\sigma = 1$, an f is required which satisfies, among other criteria, $\int_{0}^{\infty} f(u)du = 0$. As an example, $f(u) = 2e^{-2u} - 4e^{-4u}$, $u \geq 0$, yields a representation of <u>$(1 - 2^{1-s})\zeta(s)$</u> .

§2.8. In E. Grosswald [2], the authors construct functions $\zeta^*(s)$ and $L^*(s, \chi)$ which share all the complex zeros of the Riemann zeta function $\zeta(s)$ and the Dirichlet function $L(s, \chi)$, respectively. However, the analytic character of ζ^* and L^* is entirely different from that of ζ and L .

In the following we give the construction and main results of ζ^* and L^* : Let p_n be the n-th prime and select k_n so that

$p_n \le k_n \le p_{n+1}$. With these k_m form the infinite product $\zeta^*(s) = \prod_{n=1}^{\infty}(1 - k_n^{-s})^{-1}$. The product converges absolutely for $\sigma > 1$ and uniformly $\sigma \ge 1 + \varepsilon$ $(\varepsilon > 0)$, so that $\zeta^*(s)$ is holomorphic for $\sigma > 1$.

Theorem 2.1. The function $\zeta^*(s)$ possesses the following properties:

(i) $\zeta^*(s) \ne 0$ for $\sigma > 1$;

(ii) $\zeta^*(s)$ can be continued as a meromorphic function in $\sigma > 0$;

(iii) in $\sigma > 0$, $\zeta^*(s)$ has a single pole at $s = 1$ with residue r , $1/2 \le r \le 1$;

(iv) in $\sigma > 0$, $\zeta^*(s)$ has the same zeros, with the same multiplicities as $\zeta(s)$.

Let $L(s, \chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s} = \prod_p(1 - \chi(p)p^{-s})^{-1}$, where $\chi(n)$ is a Dirichlet character modulo q . For $Q \ge q$, select rational integers k_n that satisfy the two conditions $p_n \le k_n \le p_n + Q$ and $p_n \equiv k_n \pmod q$. For $\sigma > 1$ define $L^*(s, \chi)$ by the absolutely convergent infinite product $L^*(s, \chi) = \prod(1 - \chi(k)k^{-s})^{-1}$, extended over all $k = k_n$.

Theorem 2.2. The function $L^*(s, \chi)$ can be continued into the whole half plane $\sigma > 0$, where it has exactly the same zeros (including multiplicities) as $L(s, \chi)$. If $\chi(n)$ is not principal character, then $L^*(s, \chi)$ is holomorphic in $\sigma > 0$.

§2.9. In N. Levinson [5], the author gave the following identity:

(2.21) $h(s)\zeta(s) = h(s)G(s) + h(1 - s)G(1 - s)$ where

$h(s) = \Gamma(s/2)\pi^{-s/2}$, $G(s) = [(h'/_h)(s) + (h'/_h)(1 - s)]\zeta(s) + \zeta'(s)$.

In N. Levinson [7], the author considered the possibility of obtaining more general identities of this sort, by using higher derviatives of the functional equation. It is found that if P is a polynomial such that $P(x) + P(1 - x) = 1$ then we have an identity of the sort (2.21) with $G(s) \sim H(s) = \Sigma_{n \leq t/_{2\pi}} P(1 - \log n/\log(t/_{2\pi}))h^{-s}$. The original identity (2.21) corresponds to $P(x) = x$.

§2.10. Moreover, N. Levinson [14], C. Ryavec [3], K. Katayama [1], A.K. Mustafy [1] and R. Piessens; M. Branders [1] contain further remarks on the Riemann zeta function.

III. SPECIAL PROPERTIES

§3.0. In this chapter we put together those results of L-functions not related to the functional equations, nonvanishing, zeros and values at integers. We shall survey, in particular, results on mean moduli, order of magnitude and asymptotic behaviour.

The standard results in mean moduli are given in §1.2. It is well known that they are useful in proving 'Goldbach conjecture' type of theorems (cf: Pan [1]).

A result on asymptotic behaviour is that of Haviland [1]: let

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$$

$$M(r) = \max_{|s|=r} |\xi(\frac{1}{2} + is)|$$

then

$$M(r) \sim (\frac{\pi}{2})^{\frac{1}{4}}(2\pi e)^{-\frac{r}{2}} r^{\frac{r}{2}+\frac{7}{4}}.$$

as $r \to \infty$.

In §3.10, we have a result of the lower bound of the $\log \zeta(\sigma + it)$ (H.L. Montgomery [1]):

$$\text{Re } e^{-i\theta} \log \zeta(s_o) \geq \frac{1}{20} (\sigma_o - \frac{1}{2})^{\frac{1}{2}} (\log T)^{1-\sigma_o}(\log \log T)^{-\sigma_o}.$$

As usual, the author estimates an integral of the form $\int_{-\tau}^{\tau} \log \zeta(\sigma_o + it)g(t)dt$, where $g(t)$ is an positive function, then, by mean value theorem of intergation, we get an estimate of $\log \zeta(\sigma_o + it_1)$ for some t_1. The most difficult and interesting problem is to avoid the zeros of $\zeta(\sigma_o + it)$ with $|t - t_o| \leq 2\tau$, for which, the author uses a tricky lemma (§3.10 lemma 1) to solve the problem. The lemma allows us to choose "so many t_o's" such that the number of zeros in $|t - t_o| \le 2\tau$ is less than the number of the t_o's .

§3.1. We first consider the mean value theorems for Dirichlet polynomials i.e. expressions of the sort

$$\sum_{n=1}^{N} \chi(n) a_n^{-s} \tag{3.1}$$

where the a_n are arbitrary real or complex coefficients, and $s = \sigma + it$. If χ is fixed and we average with respect to t we have the very old result (Titchmarsh [1] Chap.III, IX)

$$\int_{T_o}^{T_o+t} \left|\sum_{n=1}^{N} a_n n^{-it}\right|^2 dt = (T + O(N \log N)) \sum_{n=1}^{N} |a_n|^2 \tag{3.2}$$

On the other hand, if χ is allowed to vary over all characters modulo q and s is fixed we have the equally well-known result that

$$\sum_{\chi \bmod q} \left|\sum_{n=M+1}^{M+N} \chi(n) a_n\right|^2 \le \phi(q)\left(1 + \left[\frac{-N-1}{q}\right]\right) \sum_{n=M+1}^{M+N} |a_n|^2 \tag{3.3}$$

if $N \le q$ then (3.3) holds with equality.

If we average over $q \le Q$ as well, provided that we restrict χ to primitive characters, we get:

$$\sum_{1 \le Q} \sum_{\chi}{}^{*} \left|\sum_{n=M+1}^{M+N} \chi(n) a_n\right|^2 \le (Q^2 + \pi N) \sum_{n=M+1}^{M+N} |a_n|^2 \tag{3.4}$$

Moreover, one can prove the following result: Let T_o and T be real, $T > 0$. For $1 \le r \le R$ let the numbers t_r satisfy $T_o < t_1 < t_2 < \ldots < t_R < T_o + T$ and for convenience put $T_o = t_o$ and $t_{R+1} = T_o + T$. Put $\delta = \min_{0 \le r \le R} \{t_{r+1} - t_r\}$. Then for arbitrary real or complex numbers a_n

$$\sum_{r=1}^{R}\left|\sum_{n=1}^{N} a_n n^{-t_r}\right|^2 \le (T + O(N\log N))(\delta^{-1} + \log N)\sum_{n=1}^{N}|a_n|^2 \qquad (3.5)$$

where the implied constant is absolute.

In H.L. Montgomery [3], the author combines (3.4) and (3.5) to give:

Theorem 3.1: Let N and Q be given, for each primitive character χ modulo q, $q \le Q$, let R_χ be given, and for $1 \le r \le R_\chi$ let $s_{\chi,r} = \sigma_{\chi,r} + it_{\chi,r}$ be arbitrary complex numbers. Put

$$\delta = \min_{q,\chi,a\neq b} |t_{\chi,a} - t_{\chi,b}| \quad , \qquad (A)$$

$$T = 1 + \max_{q,\chi,r} t_{\chi,r} - \min_{q,\chi,r} t_{\chi,r} \quad , \qquad (B)$$

and finally

$$\sigma = \min_{q,\chi,r} \sigma_{\chi,r} \quad . \qquad (C)$$

Then

$$\sum_{q\le Q}\sum_{\chi}{}^{*}\sum_{r=1}^{R_\chi}\left|\sum_{n=1}^{N}\chi(n)a_n n^{-s_{\chi,r}}\right|^2$$

$$\ll (Q^2T + N)\left(1 + \frac{(\log N)^2}{\delta}\right)(\log N)^4\sum_{n=1}^{N}|a_n|^2 n^{-2\sigma} \quad .$$

If $R_\chi = 1$ and $s_{\chi,1} = 0$ for all χ then $T = 1$ and we have a slightly weaken form of (3.4). If $Q = 1$ then we have a weakened form of (3.5).

The author also gave two examples of the strong results he can obtain concerning the large values taken on by (3.1).

Theorem 3.2: Under the same hypotheses as in Theorem 1, if V is so large that

$$V^2 \ge A Q T^{\frac{1}{2}}(\log QT)^{\frac{1}{2}}(\log \operatorname{Log} N)\sum_{n=1}^{N}|a_n|^2 n^{-2\sigma}$$

where A is a large absolute constant, then the number of triples (q, χ, r)

with $q \le Q$, χ primitive modulo q , and $1 \le r \le R_\chi$ such that

$$\left|\sum_{n=1}^{N} \chi(n) a_n n^{-s_{\chi,r}}\right| \ge V$$

is
$$\ll \frac{N}{V^2}\left(1 + \frac{\log N}{\delta}\right) \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma} .$$

Theorem 3.3. For $1 \le r \le R$ let $s_r = \sigma_r + it_r$ be given, and suppose that (A), (B), and (C) hold, where now $q = 1$, $\chi = \chi_0$. Let

$$M(\alpha, T) = \max_{\substack{\sigma \ge \alpha \\ |t| \le T \\ |s-1| \ge 1}} |\zeta(s)|$$

If V is so large that $V^2 \ge AM(\frac{1}{2}, 2T)N^{\frac{1}{2}} \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma}$ where A is a large absolute constant, and if

$$\left|\sum_{n=1}^{N} a_n n^{-s_r}\right| \ge V \quad \text{for } r = 1, 2, \ldots, R , \text{ then}$$

$$R \ll \frac{N}{V^2}\left(1 + \frac{\log N}{\delta}\right) \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma}$$

The proofs of Theorem 3.1, 3.2 and 3.3 are based on the following

Lemma For $M + 1 \le n \le M + N$ let a_n be arbitrary real or complex numbers, and for $1 \le j \le J$ let χ_j be any character, $s_j = \sigma_j + it_j$ any complex number. Let

$$\sigma = \min_j \sigma_j .$$

For all positive n let $f(n)$ be real and subject to the following conditions:

a) $f(n) \ge 0$ for all n

b) $f(n) \ge 1$ for $M + 1 \le n \le M + N$,

and put $F = F(f) = \sum_{n=1}^{\infty} f(n)$.

Then

$$\sum_{j=1}^{J} \left| \sum_{n=M+1}^{M+N} \chi_j(n) a_n n^{-s_j} \right| \le (F^{\frac{1}{2}} J^{\frac{1}{2}} + K^{\frac{1}{2}} J) \left(\sum_{n=M+1}^{M+N} |a_n|^2 n^{-2\sigma} \right)^{\frac{1}{2}}$$

where $\quad K = \max_{j \neq k} \left| \sum_{n=1}^{\infty} f(n) \chi_j(n) \overline{\chi}_k(n) n^{2\sigma - s_j - \overline{s_k}} \right| \quad .$

A simple argument will give us the following very useful

Corollary: Under the same hypotheses, if

$$V^2 \ge 4K \sum_{n=M+1}^{M+N} |a_n|^2 n^{-2\sigma}$$

and if for $1 \le j \le J$

$$\left| \sum_{n=M+1}^{M+N} \chi_j(n) a_n n^{-s_j} \right| \ge V \quad ,$$

then

$$J \le \frac{4F}{V^2} \sum_{n=M+1}^{M+N} |a_n|^2 n^{-2\sigma} \quad .$$

For some suitable choices of $f(n)$ in the lemma, we can prove the theorems. In the proof of Theorem 3.1, the author put $f(u) = 5(e^{-(\frac{n}{2N})^k} - e^{-(\frac{n}{N})^k})$ and in the proofs of Theorem 3.2 and 3.3, he put $f(n) = e^{1 - \frac{n}{N}}$.

§3.2. The fourth moment of $\zeta(\frac{1}{2} + it)$ is given by

$$\sum_{r=1}^{R} |\zeta(\tfrac{1}{2} + it_r)|^4 \ll (\delta^{-1} + \log T) T \log^4 T \tag{3.6}$$

where $\qquad t_1 < t_2 < \dots < t_R \ , \quad |t_r| \le T \ , \qquad T \ge 2$

and $\qquad \delta = \min_r (t_{r+1} - t_r)$

K. Ramachandra generalized this to

$$\sum_{r=1}^{R} \left| \frac{\zeta(\sigma + itr)}{C_1 \log tr} \right|^{\frac{2}{1-\sigma_r}} < T \log^4 T (\log T + \delta^{-1})$$

where $\frac{1}{2} \leq \sigma_r < 1$, $T \geq 30$, $t_o = T < t_1 \quad \dots < t_R < 2T = t_{R+1}$.

$\delta = \min_r (t_{r+1} - t_r)$ and C_1 is an absolute positive constant. In K. Ramachandra [4] the author consider a somewhat extension of the above result in a certain direction. Assume

$$|\zeta(s)| < C_2 t^{d(1-\sigma)^{3/2}} (\log t)^{2/3} \quad \text{with } d = 100 ,$$

the author proved the following theorem:

Theorem 3.4.

Let $\frac{1}{2} \leq \sigma_r < 1$, $T \geq 30$, $t_o = T < t_1 < \dots < t_R < 2T = t_{R+1}$, $\delta = \min_r (t_{r+1} - t_r)$, $a = a^{-5/2} d^{-1}$, $F(\sigma_r) = \max\left(\frac{2}{1-\sigma_r}, \frac{4a}{3(1-\sigma_r)^{3/2}}\right)$, and

$G(\sigma_r, t_r) = (C(1-\sigma_r)^{-3/2} \log t_r)^{\alpha}$ where $\alpha = 2^{-1}(1-\sigma_r)^{-3/2} a + 5$,

and C is a suitable absolute positive constant. Then

$$\sum_{r=1}^{R} \frac{\zeta(\sigma + it_r)}{G(\sigma_r, t_r)}^{F(\sigma_r)} < T(1 + \delta^{-1}) .$$

Moreover, a corollary to the theorem is stated:

Corollary. With the notation stated in the theorem,

$$\sum_{r=1}^{R} \left| \frac{\zeta(\sigma_r + it_r)}{C_3 \exp(\sqrt{\log t_r \log\log t_r})} \right|^{F(\sigma_r)} < T(1 + \delta^{-1})$$

where C_3 is an absolute positive constant.

§3.3. Let χ be a Dirichlet character (mod q) , t a real number, and set $\tau = |t| + 2$, then A. Fujii [4] proved that

(i) $\quad \Sigma_{n \leq N} \chi(n) n^{it} = \varepsilon(\chi)[\phi(q) N^{1+it}/q(1 + it)] + O((q\tau)^{\frac{1}{2}} \log q\tau)$

(ii) $$\sum_{n\le N}(1 - (n/N))\chi(n)n^{it} = \varepsilon(\chi)[\psi(q)N^{1+it}/q(1 + it)(2 + it)] + O((q\tau)^{\frac{1}{2}} \log q\tau)$$

where $\varepsilon(\chi) = 1$ or 0 according to whether χ is principal or not, and relations of the estimates of these sums to hybrid estimates of $L(\frac{1}{2} + it, \chi)$ are studied. In addition, it is proved that if P is a finite set of primes and q is composed only of primes in P, then $L(\frac{1}{2} + it, \chi) \ll_{\varepsilon,P} (q\tau)^{1/6+\varepsilon}$.

§3.4. Let χ be the Dirichlet character $(\text{mod } |q|)$ determined by Kronecker's symbol $(\frac{q}{n})$. Here q is assumed to be a non-square integer satisfying $q \equiv 0$ or $1 \pmod 4$. A summation over such values of q will be denoted by Σ', then, M. Jutila [4] proved the following theorems:

Theorem 3.5. For $X \ge 3$, $T > 0$, we have

$$\sum_{|q|\le X}' \int_{-T}^{T} |L(\tfrac{1}{2} + it, \chi)|^2 dt \ll XT \log^{16}(X(T + 1)) .$$

Theorem 3.6. For $X \ge 3$ and $1 \le N \le Y$ we have

$$\sum_{|q|\le X}' \sum_{1\le n\le Y} \Big| \sum_{0\le\nu\le N} \chi(n + \nu)\Big|^2 \ll XYN \log^{17} X .$$

§3.5. Consider the error term $E(T)$ in the well-known mean value theorem $\int_0^T |\zeta(\frac{1}{2} + it)|^2\, dt = T \log(T/_{(2\pi)}) + T(2\gamma - 1) + E(T)$, $T \ge 2$. The function $E(T)$ has been estimated by various authors. In A. Good [1], the author gave a new proof for $E(T) = O(T^{\alpha})$, where $\alpha > 27/_{82}$, which has been proved by R. Balasubramanian. In this paper $E(T)$ is represented by two infinite double series and an error term of $O(1)$, and in A. Good [2], the author proved that $E(T) = \Omega(T^{\frac{1}{4}})$ as $T \to \infty$. This is deduced from a theorem estimating

$$\int_0^X dT\{\int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^2 dt - (T+H)\mathrm{lot}(T+H) + T\log T + H(1+\log 2\pi - 2\gamma)\}^2$$

as equal to a complicated sum $I(X, H)$ + an error term as $X \to \infty$ uniformly for $1 \le H \le [(2X/\pi)^{\frac{1}{2}}]$.

§3.6. While D.R. Heath-Brown [3] established

$$\int_2^T E^2(t)dt = (\tfrac{2}{3})(2\pi)^{-\frac{1}{2}}\zeta^4(\tfrac{2}{3})(\zeta(3))^{-1}T^{3/2} + O(T^{5/4}(\log T)^2)$$

using Atkinson's formula $E(t) = A(t) + B(t) + O((\log t)^2)$ where A and B are certain sums involving the divisor function.

§3.7. K. Ramachandra [6] gave a new elegant method for proving estimates of the mean fourth power estimate for $\zeta(\frac{1}{2}+it)$ and $L(\frac{1}{2}+it, \chi)$. The idea is to find for $L^2(s, \chi)$ an expression of the form $\Sigma_{n=1}^{\infty}\tau(n)\chi(n)e^{-n/N}n^{-s} + I_1 + I_2$ + small error. where I_1 and I_2 are certain contour integrals involving the segments $\Sigma_{n=1}^{N}$ and $\Sigma_{n=N+1}^{\infty}$ if the Dirichlet series for $L^2(s, \chi)$.

§3.8. K. Ramachandra [7], the author sketched a new proof of the following mean value theorem of A.E. Ingham [6]. For $T \ge 2$,

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt = \tfrac{1}{2}\pi^{-2}T\log^4 T + O(T\log^3 T) .$$

§3.9. In D.R. Heath-Brown [4], the author proved that $\int_0^T |\zeta(\frac{1}{2}+it)|^{12} \ll T^2(\log T)^{17}$, if $k \ge 15$, the author showed that

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^k dt \ll T^{1+173(k-6)/1067}(\log T)^{2k} .$$

§3.10. Ω-theorems

If the Riemann hypothesis is true, then

$$\overline{\lim_{t\to\infty}} \frac{|\zeta(1+it)|}{\log\log t} = \lambda \le 2e^{\gamma}$$

$$\overline{\lim_{t\to\infty}} \frac{1/|\zeta(1+it)|}{\log\log t} = \mu \le \frac{12}{\pi^2} e^{\gamma}$$ where γ is Euler's constant.

E.C. Titchmarsh [1] stated and proved two results of Littlewood: (see Titchmarsh [1] Theorem 8.9(A), 8.9(B)).

Theorem 3.7. $$\overline{\lim_{t\to\infty}} \frac{|\zeta(1+it)|}{\log\log t} \ge e^{\gamma} .$$

Theorem 3.8. $$\overline{\lim_{t\to\infty}} \frac{1/|\zeta(1+it)|}{\log\log t} \ge \frac{6}{\pi^2} e^{\gamma} .$$

Thus on the Riemann hypothesis it is only a factor 2 which remains in doubt in each case.

In N. Levinson [9], the author slightly improved the above results as follows:

Theorem 3.9. There exist arbitrarily large t for which
$|\zeta(1+it)| \ge e^{\gamma} \log\log t + O(1)$.

Theorem 3.10. There exist arbitrarily large t for which
$$\frac{1}{|\zeta(1+it)|} \ge \frac{6e^{\gamma}}{\pi^2} (\log\log t - \log\log\log t) + O(1) .$$

In N. Levinson [9], the author also gave the following theorem:

Theorem 3.11. Let $\frac{1}{2} \leq \sigma < 1$. Then there exist a positive constant B independent of σ such that for sufficient large T

$$\max_{1 \leq |t| \leq T} |\zeta(\sigma + it)| \geq e^{B(\log T)^{1-\sigma}/\log\log T} .$$

This improves slightly the result of Titchmarsh [1]. Theorem 8.12. Moreover, for $\frac{1}{2} < \sigma < 1$, H.L. Montgomery [1] gave a sharper lower bound than is provided by Theorem 3.11. The author showed that $|\zeta(s)|$ becomes correspondingly small, and that $\arg \zeta(s)$ becomes correspondingly large in both signs. We write $s = \sigma + it$.

Theorem 3.12. Let $\frac{1}{2} < \sigma_o < 1$, $T > T_o(\sigma_o)$. For any real θ there is a t_o such that $T^{(\sigma_o - 1/2)/3} \leq t_o \leq T$ and

$$\operatorname{Re} e^{-i\theta} \log \zeta(s_o) \geq \frac{1}{20} (\sigma_o - \frac{1}{2})^{\frac{1}{2}} (\log T)^{1-\sigma_o} (\log\log T)^{-\sigma_o} .$$

Taking $\theta = 0, \pi, \pm\pi/2$ in the above, the author derived the following

Corollary. Let σ be fixed, $\frac{1}{2} < \sigma < 1$. Then as t tends to infinity,

$$\log|\zeta(s)| = \Omega_{+}((\log t)^{1-\sigma}(\log\log t)^{-\sigma}) ,$$

and

$$\arg \zeta(s) = \Omega_{\pm}((\log t)^{1-\sigma}(\log\log t)^{-\sigma}) .$$

The proof of theorem 3.12 is based on the following lemmas:

Lemma 1. Let $\theta_1, \ldots, \theta_M$ be arbitrary real numbers, and suppose that $0 < \delta < \frac{1}{2}$. There are at least $[\delta^M(R + 1)]$ integers r such that $1 \leq r \leq R$ and $||r\theta_m|| \leq \delta$ for $1 \leq m \leq M$.

Here $||\theta||$ denotes the distance from θ to the nearest integer,

$$||\theta|| = \min_{n \in Z} |\theta - n|$$

(cf: Cassels [1] chap. III Th.2 p.71).

Lemma 2. Suppose that $\frac{1}{2} \leq \sigma_o < 1$, $t_o \geq 15$, and that $\zeta(s) \neq 0$ for $\sigma > \sigma_o$, $|t - t_o| \leq 2\tau$, where $\tau = \tau(t_o) = (\log t_o)^2$. Then for $\alpha > 0$, and real h,

$$\frac{2}{\pi} \int_{-\tau}^{\tau} \log \zeta(s_o + it) \left(\frac{\sin \alpha t}{t}\right)^2 e^{iht} dt = \Sigma \Lambda_1(n) w_n n^{-s_o} + O(e^{|n|+2\alpha} (\log t_o)^{-2}) \quad (3.7)$$

where $\Lambda_1(n) = \dfrac{\Lambda(n)}{\log n}$ $(\Lambda_1(1) = 0)$

and $w_n = \max(0, \alpha - |h - \log n|)$.

Lemma 3. For $T \geq 10$, $\frac{1}{2} \leq \sigma \leq 1$,

$$N(\sigma, T) << T^{3/2-\sigma} (\log T)^5$$

where $N(\sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of the zeta function $\zeta(s)$ for which $\beta \geq \sigma$, $0 \leq \gamma \leq T$.

Lemma 4. If $x \geq 1$, $\frac{1}{2} < \sigma_o < 1$, then

$$\sum_{|\log \frac{n}{x}| \leq \frac{1}{2}} \Lambda_1(n) n^{-\sigma_o} \left(\frac{1}{2} - |\log \frac{n}{x}|\right) = \left[\left(\frac{2 \sin h(\frac{1-\sigma_o}{4})}{1 - \sigma_o}\right)^2 + o(1)\right] \frac{x^{1-\sigma_o}}{\log x}$$

$$\geq \left[\frac{1}{4} + o(1)\right] \frac{x^{1-\sigma_o}}{\log x} \quad (3.8)$$

Let $\alpha = \frac{1}{2}$, and take successively $h = -\log x$, $h = 0$, $h = \log x$, where $x \geq 1$ in (3.7), for the three respective values of h the atuhor multiplies (3.7) by $\frac{1}{2}e^{-i\theta}$, 1, $\frac{1}{2}e^{i\theta}$, and sum, to find that

$$\frac{2}{\pi}\int_{-\tau}^{\tau} \log \zeta(s_o + it)\left(\frac{\sin t/2}{t}\right)^2 (1 + \cos(\theta + t \log x))dt$$

$$= \frac{1}{2}e^{-i\theta} \sum_{|\log \frac{n}{x}| \le \frac{1}{2}} \Lambda_1(n)n^{-s_o}\left(\frac{1}{2} - \left|\log \frac{n}{x}\right|\right) + O(x(\log t_o)^{-2}) \qquad (3.9)$$

provided that $\zeta(s) \neq 0$ for

$$\sigma \ge \sigma_o \,, \quad |t - t_o| \le 2(\log t_o)^2 \qquad (3.10)$$

Since $\frac{2}{\pi}\int_{-\tau}^{\tau} \left(\frac{\sin t/2}{t}\right)^2 (1 + \cos(\theta + t \log x))dt \le \frac{4}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\sin t/2}{t}\right)^2 dt = 1$, and the first integrand is non-negative, then, by the mean value theorem of integration and (3.9) we obtain

$$\operatorname{Re} e^{-i\theta}\log \zeta(\sigma_o + it_1) \ge \operatorname{Re} \sum_{|\log \frac{n}{x}| \le \frac{1}{2}} \Lambda_1(n)n^{-s_o}\left(\frac{1}{2} - \left|\log \frac{n}{x}\right|\right) + O(x(\log t_o)^{-2}). \qquad (3.11)$$

where $|t_1 - t_o| \le 2(\log t_o)^2$.

By lemma 4, we can estimate $\operatorname{Re} \sum_{|\log \frac{n}{x}| \le \frac{1}{2}} \Lambda_1(n)n^{-\sigma_o}\left(\frac{1}{2} - \left|\log \frac{n}{x}\right|\right)$, such that, if $\operatorname{Re} n^{-it_o} \ge \frac{1}{2}$, we have

$$\operatorname{Re} \sum_{|\log\frac{n}{x}| \le \frac{1}{2}} \Lambda_1(n)n^{-s_o}\left(\frac{1}{2} - \left|\log \frac{n}{x}\right|\right) \ge \left(\frac{1}{8} + o(1)\right)\frac{x^{1-\sigma_o}}{\log x} \qquad (3.12)$$

for suitable value of x , (3.11) gives theorem 3.12, where lemma 1 and lemma 3 guarantee that we can choose a t_o such that $\zeta(s_o + it_1) \neq 0$ in (3.10).

N. Levinson [8], [10] gave more results about the Ω-theorems for the Riemann zeta function.

§3.11. Let χ be a character (mod q), in D.R. Heath-Brown [2] the author consider the order of magnitude of $L(s, \chi)$ along the critical line $Re(s) = \frac{1}{2}$. The trivial bound in the context is

$$L(\frac{1}{2} + it, \chi) << (qT)^{\frac{1}{4}} \tag{3.13}$$

where $T = |t| + 1$.

D.A. Burgess [2] has given bounds for $L(s, \chi)$ that are sharper than (3.13) with respect to q; although he does not give the dependence on T explicity, it is clear that his method yields

$$L(\frac{1}{2} + it, \chi) << q^{3/16 + \varepsilon} T, \tag{3.14}$$

for any $\varepsilon > 0$, (3.14) is weaker than (3.13) for $q \leq T^{12}$. Moreover, the following bound

$$L(\frac{1}{2} + it, \chi) << q^{1/2} T^{1/6} (\log qT) \tag{3.15}$$

is weaker than (3.13) for $q \geq T^{1/3}$; thus neither (3.14) nor (3.15) improves upon (3.13) for the intermediate rang $T^{1/3} < q < T^{12}$. The object of D.R. Heath-Bronw [2] is to prove the following estimates which, together, supersede (3.13), (3.14) and (3.15) for all values of q and T. The generalized Burgess method gives

Theorem 3.13. $L(\frac{1}{2} + it, \chi) << d(q) q^{3/16} T^{1/4} (\log q)^2$. The generalized van der corput method gives

Theorem 3.14. Let $q_0 | q$, then

$$L(\frac{1}{2} + it, \chi) << (d(q))^3 (q_0^{1/2} + (q/q_0)^{1/4} (\log qT)^{1/2} + (qT)^{1/6}) (\log qT).$$

Here, $d(q)$ denotes the number of divisors of q. We note that (3.14) is superseded by Theorem 3.13, and (3.15) by Theorem 3.14 with $q_0 = 1$. The corollary below improves upon (3.13) for all values of q and T.

Corollary 1. $$L(\tfrac{1}{2} + it, \chi) \ll (qT)^{1/5}(\log q)^2 .$$

Finally, from Theorem 3.14, we may deduce an estimate without the factor $(d(q))^3$.

Corollary 2. $$L(\tfrac{1}{2} + it, \chi) \ll (q^{\frac{1}{4}}(\log q)^{\frac{1}{2}} + (d(q))^3(qT)^{1/6}) \times (\log qT)$$

§3.12. We consider L-functions in the half-plane $\sigma > \frac{1}{2}$. In P.D.T.A. Elliott [1], the author determined the average of $|L(s, \chi)|^2$ over all non-principal characters of prime moduli $p \le Q$, for fixed s. The author also showed that the proportion of these characters for which $|\varepsilon(\chi)L(s, \chi)| \le z$ converges to a distribution function as $Q \to \infty$. Here s is a fixed point in the half plane $\sigma > \frac{1}{2}$, and $\varepsilon(\chi) = \varepsilon(s, \chi)$ satisfies $0 < c_1 < |\varepsilon(\chi)| < c_2 < \infty$, uniformly in $\sigma \ge \frac{1}{2} + \delta$. In P.D.T.A. Elliott [2], the following theorem was given:

$v_Q(z, s) = M_Q^{-1} \sum_{p \le Q} \sum_{\chi \ne \chi_0, \arg L(s, \chi) < z (\mathrm{mod}\ 1)} 1$, where

$M_Q = \sum_{p \le Q}(p - 1)$, $s = \sigma + it$ and $\sigma > \frac{1}{2}$; then $v_Q(z, s)$ tends to a continuous distribution function as $Q \to \infty$. The characteristic function of this distribution is of the form $\psi(k) = \prod_p (1 + \sum_{m=1}^{\infty} \binom{-k/2}{m}\binom{k/2}{m}) p^{2m\sigma}$, where $k = 0, \pm 1, \pm 2, \ldots$. The proof vests on methods from analytic number theory and on limit-theorems from probability theory.

§3.13. If s is an arbitrary complex number in the half-plane $\sigma > \frac{1}{2}$,

E. Stankus [2] associated a sequence P_Q of Borel measures on the complex plane defined by

$$P_Q(A) = (2 \log Q/_Q 2)\Sigma_{p\ Q} N\{X(\text{mod } p) : \chi \neq \chi_o, L(s, \chi) \in A\} \quad .$$

and showed that it converges weakly to a certain measure.

§3.14. In A. Good [3], the author studied the behaviour of the discrete mean $(1/K)\Sigma_{k=1}^{K} |F(\sigma + i\beta + ik\alpha)|^2$, $K \to \infty$, for a large class of Dirichlet series F .

§3.15. Let $f_{k,m}(n)$ be defined by the equation $\{\zeta(s)/\zeta(ks)\}^m = \Sigma_{n=1}^{\infty} f_{k,m}(n)n^{-s}$, where $k \geq 2$ and m are positive integers and $\zeta(s)$ is the Riemann zeta function. A. Ivic [1] obtained an asymptotic formula for the sum $\Sigma_{n \leq x} f_{k,m}(n)$ and showed that the error term is related to the corresponding problem for the numbers generated by $(\zeta(s))^m$.

IV. $L(1, \chi)$ & NONVANISHING REGIONS

§4.0. The most famous nonvanishing theorems are $L(1, \chi) \neq 0$ and $L(s, \chi) \neq 0$ for $\sigma > 1$. One improves these results by obtaining better bounds for $L(1, \chi)$ or by extending the nonvanishing region into critical strip $0 < \sigma < 1$ (modulo the possible existence of an 'exceptional zero' on the real axis).

In J. Pintz [2] I, the author used a lemma of real elementary analysis to simplify the proof of his theorem. We slightly change a value in the original lemma and gave a proof to the "new" one. (see §4.14) (We set
$$\sum_{m\le u} \frac{1}{m^{1-\tau}} = \frac{1}{\tau}(u^{\tau} - 1) + c_{\tau} + \frac{\theta}{[u]^{1-\tau}}$$
instead of
$$\sum_{m\le u} \frac{1}{m^{1-\tau}} = \frac{1}{\tau}(u^{\tau} - 1) + c_{\tau} + \frac{\theta}{u^{1-\tau}}) .$$

§4.1. Lower bound for $L(1, \chi)$.

4.1.1 Siegel [1] in 1935 proved that for every ε there exists a $c(\varepsilon) > 0$ such that
$$L(1, \chi) > c(\varepsilon)q^{-\varepsilon}$$
for real primitive character χ mod q. The proof was simplified by Estermann (1948) and Chowla (1950). An important elementary proof was given by Linnik [1] in 1950. Making use of the multiplicativity of Liouville's λ function Pintz [3] was able to further simply Linnik's proof.

4.1.2 If we assume that q is greater than a given effective constant q_o and an L-function belonging to a nonprincipal (real or complex) character χ_k mod k has an zero $s_o = 1 - \gamma + it$ with $\gamma < 0.05$. Then Pintz [2] IV showed that for an arbitrary real non-principal character

χ_q mod q (for which $\chi_k\chi_q$ is also non-principal) the inequality

$$L(1, \chi_q) > \frac{1}{140\, u^{6\gamma} \log^3 u}$$

holds, where $u = k|s_o|q$.

4.1.3 If χ is a real primitive character mod q then Joly and Moser [1] showed that if $L'(1, \chi) < \frac{\pi^2}{6} - \varepsilon$ there is a calculable constant $c(\varepsilon)$ such that

$$\frac{c(\varepsilon)}{\log q} < L(1, \chi)$$

and S. Chowla [2] showed that if hypothesis (ζ) is true, and $q \equiv 3 \pmod 4$ there is a constant c such that $\frac{c}{\log q} < L(1, \chi)$.

4.1.4 In J. Pintz [2]I, the author gave an elementary proof of Hecke's theorem:

Theorem. If an L-function belonging to the real non-principal character $\chi \pmod q$ (where $q \geq 200$) has no zero in the interval $[1 - \alpha, 1]$, where $0 < \alpha < (20 \log q)^{-1}$, then

$$L(1, \chi) > 0.23\alpha .$$

The proof is a simple calculation based on the following lemma:

Lemma. For an arbitrary τ , for which $0 < \tau < 1$, there exists a c_τ , $0 < c_\tau < 1$, such that for all $u \geq 1$.

$$\sum_{m \leq u} \frac{1}{m^{1-\tau}} = \frac{1}{\tau}(u^\tau - 1) + c_\tau + \frac{\theta}{[u]^{1-\tau}} \qquad (|\theta| \leq 1) .$$

We first state a simpler result which can be found in L.K. Hua [1] p.101:

If $x \ge a$, $f(x)$ is an positive decreasing function, then

$$\lim_{N\to\infty}\left(\sum_{n=a}^{N} f(n) - \int_a^N f(x)dx\right) = \alpha \quad \text{exist}$$

and $0 \le \alpha \le f(a)$.

To prove the lemma, we let $f(x) = \dfrac{1}{x^{1-\tau}}$, $a = 1$. set

$$g(u) = \sum_{1\le n\le u} f(n) - \int_1^u f(x)dx \ , \quad \text{then}$$

$$\begin{aligned}
g(u) - \alpha &= \sum_{1\le n\le u} f(n) - \int_1^u f(x)dx - \lim_{N\to\infty}\left(\sum_{n=1}^{N} f(n) - \int_1^N f(x)dx\right) \\
&= \sum_{n=1}^{[u]} f(n) - \int_1^{[u]} f(x)dx - \int_{[u]}^{u} f(x)dx \\
&\qquad - \lim_{N\to\infty}\left(\sum_{n=1}^{N} f(n) - \int_1^N f(x)dx\right) \\
&= -\int_{[u]}^{u} f(x)dx - \lim_{N\to\infty}\left(\sum_{n=[u]+1}^{N} f(n) - \int_{[u]}^{N} f(x)dx\right) \\
&= -\int_{[u]}^{u} f(x)dx + \lim_{N\to\infty} \sum_{n=[u]+1}^{N} \int_{n-1}^{n} (f(x) - f(n))dx
\end{aligned}$$

$$\begin{cases}
\le \lim\limits_{N\to\infty} \sum\limits_{n=[u]+1}^{N} \int_{n-1}^{n} (f(u-1) - f(u))dx = \lim\limits_{N\to\infty}(f([u]) - f(N)) \\
\ge -\int_{[u]}^{u} f(x)dx \ge -(u - [u])f([u]) \ge -f([u]) \quad .
\end{cases}$$

Since $f(x) = \dfrac{1}{x^{1-\tau}} \Rightarrow \lim\limits_{x\to\infty} f(x) = 0$.

$$\therefore \qquad -f([u]) \le g(u) - \alpha \le f([u])$$

$$\Rightarrow \ |g(u) - \alpha| \le f([u])$$

$$\Rightarrow \ g(u) - \alpha = \theta f([u]) \qquad \text{where} \quad |\theta| \le 1 \ .$$

this proves the lemma.

§4.2. Upper bound for $L(1, \chi)$

4.2.1 If χ is a real nonprincipal character (mod q) , then the upper bound which one can give for $L(1, \chi)$ is closely connected with the upper bound of $S = \max_{1 \le a \le b \le q} \left| \sum_{n=1}^{b} \chi(n) \right|$.

Using the trivial $S_\chi \le q$ one can easily prove $L(1, \chi) \le \log q + O(1)$; by means of the Polya-Vinogradov inequality $S_\chi \le c\sqrt{q} \log q$, Polya [1] prove that

$$L(1, \chi) \le (\frac{1}{2} + O(1))\log q .$$

4.2.2 If $q = p$ is a prime, χ_p a real nonprincipal character (mod p), S. Chowla [4] in 1964 proved the inequality

$$L(1, \chi_p) \le (\frac{1}{4} + o(1))\log p .$$

D.A. Burgess [1] showed in 1966 that

$$L(1, \chi_p) < 0.2456 \ldots \log p .$$

Wirsing (unpublished) improved it to

$$L(1, \chi_p) < \frac{1}{2}(\sqrt{2} - 1 + o(1)) \log p \approx 0.207 \log p .$$

P.J. Stephens [1] showed in 1972 using a method of Wirsing that

$$L(1, \chi_p) < \frac{1}{2}(1 - \frac{1}{\sqrt{e}} + o(1)) \log p \approx 0.197 \log p .$$

4.2.3 In J. Pintz [2]VII, the author gave an elementary proof of Stephen's result generalizing it for real primitive characters, whose modulus is not necessarily prime. The result will follow from the following general theorem.

Theorem. If θ is a completely multiplicative function, which takes only the values $+1, 0, -1$, x a real number for which

$$\sum_{n\le x}\theta(n) \le \varepsilon x$$

then $$\sum_{d\le x}\frac{\theta(d)}{d} \le 2\left(1-\frac{1}{\sqrt{e}}+\delta\right)\log x$$

where $\delta = \delta(\varepsilon, x) \to 0$ if $x \to \infty$ and $\varepsilon \to 0$.

Using the result of D.A. Burgess [3]: if χ is a nonprincipal character (mod q) , then

$$\left|\sum_{n=N+1}^{N+H}\chi(n)\right| \le \varepsilon H \quad \text{for } H \ge q^{\tau_\chi+\varepsilon}, \quad q > q_o(\varepsilon)$$

where if χ is a primitive character then $\tau_\chi = \frac{1}{4}$ and for an arbitrary χ , $\tau_\chi = 3/8$, we have

$$\left|\sum_{q^{\tau_\chi+\varepsilon}<d\le q}\frac{\chi(d)}{d}\right| \le \varepsilon \log q$$

and using $S_\chi \le q$ by means of Abel's inequality we get

$$\left|\sum_{d>q}\frac{\chi(d)}{d}\right| \le 1 .$$

So using the above Theorem with $x = q^{\tau_\chi+\varepsilon}$ we have theorem: If χ is a real primitive character (mod q),

then $$L(1, \chi) \le \frac{1}{2}\left(1-\frac{1}{\sqrt{e}}+o(1)\right)\log q .$$

And if χ is a real nonprincipal character (mod q),

then $$L(1, \chi) \le \frac{3}{4}\left(1-\frac{1}{\sqrt{e}}+o(1)\right)\log q .$$

4.2.4 I.M. Vinogradov conjectured more than 50 years ago, that the least prime quadratic residue mod p (p is a prime).

$P(p) < c(\varepsilon)p^{\varepsilon}$ where ε is an arbitrary positive number and $c(\varepsilon)$ a constant depending on ε .

Conditional results connecting the hypothesis of I.M. Vinogradov with the value of $L(1, \chi_p)$ - where $\chi_p(n) = (\frac{n}{p})$ - were achieved by Yu. V. Linnik and A. Renyi [1], P.D.T.A. Elliott [3] and D. Wolke [1]:

Linnik and Renyi showed that if $P(p) > p^{1/k}$ then

$L(1, \chi_p) = \sum_{n=1}^{\infty} \frac{(\frac{n}{p})}{n} << 1$. On the same condition, Elliott proved

$$L(1, \chi_p) << \frac{(\log \log p)^k}{\log p} .$$

Wolke proved

$$L(1, \chi_p) << \frac{k^2}{\log p} .$$

J. Pintz [2]VI. gave a simple, elementary proof for Wolke's result:

Theorem : If the least prime quadratic residue

(mod p) $\qquad P(p) > p^{\varepsilon} \geq P_o$.

where $\varepsilon \leq \frac{1}{2}$, P_o is an absolute constant, then the inequality

$L(1, \chi_p) = \sum_{n=1}^{\infty} (\frac{n}{p}) n^{-1} \leq \frac{24}{\varepsilon^2 \log P}$ holds.

The proof is based on the following lemma:

Lemma: If χ is a real non-principal character mod q, $x > \sqrt{q} \log^2 q$, $g(n) = \sum_{d|n} \chi(d)$, then the equality

$\sum_{n \leq x} \frac{g(n)}{n} = L'(1, \chi) + L(1, \chi)(\log x + c) + O(\sqrt{\frac{\sqrt{q} \log q \log x}{x}})$ holds, where c denotes Euler's constant.

4.2.5 Many results in the literature are proved under various hypotheses on Dirchlet L-function concerning, e.g., zero-free regions for $L(s, \chi)$ or the magnitude of $L(1, \chi)$, or the least prime Y_+, Y_- such that $\chi(Y_+) = 1$, $\chi(Y_-) = -1$ respectively, Friedlander [1] discussed some connections between these assumptions.

Let q be an odd prime, $\chi(n)$ the Legendre symbol $(\frac{n}{q})$. Various results are known connecting together:

(A) zero-free regions for $L(s, \chi)$

(B) the magnitude of $L(1, \chi)$

(C) the magnitudes of Y_+ and Y_- .

Roughly speaking, a statement about any of these implies a corresponding statement about the subsequent ones. In J.B. Friedlander [1], the author concerned with results of the type (B) $\Rightarrow$ (C) primarily. The first result of this nature seems to be that of Yu. V. Linnik and A. Renyi [1]:

Given $\varepsilon > 0$, there exists $c(\varepsilon)$ such that, if

$$L(1, \chi) < c(\varepsilon) \log q \text{ ; then } Y_- < q^{\varepsilon} \quad .$$

More recently, D. Wolke [1] gave a somewhat analogous result for Y_+ :

$$\log Y_+ << (\frac{\log q}{L(1, \chi)})^{\frac{1}{2}} \quad .$$

J.B. Friedlander [1] proved Wolke's result in a different approach which can be outlined as follows: Let $g(m)$ and $h(m)$ be totally multiplicative functions having absolute value ≤ 1 . One might expect that if $\sum_{m \leq X} \frac{g(m)}{m}$ is in some sense small, then so is $\sum_{\substack{m \leq X \\ p | m \to p \leq Y}} \frac{g(m)}{m}$ and conversely. Hence, if $h(m) = g(m)$ for all $m \leq Y$, then an upper bound for $\sum_{m \leq X} \frac{g(m)}{m}$ should lead to an upper bound for $\sum_{n \leq X} \frac{h(m)}{m}$.

Moreover, the author also gave some conjectures on the magnitude of $\log Y_-$.

§4.3. Bound for $|L(1, \chi)|$.

4.3.1 First of all, if χ denotes the Legendre symbol modulo p , S. Chowla [1] proved that

$$|L(1, \chi)| < (\frac{1}{4} + \varepsilon)\log p$$

for $p > p_0(\varepsilon)$. D.A. Burgess [1] improved the right hand member of this inequality to $.2456 \log p$ and P.J. Stephens [1] proved that

$$|L(1, \chi)| < \{\frac{1}{2}(1 - \frac{1}{\sqrt{e}}) + \varepsilon\}\log p$$

for $p > p_0$. On the other hand, D. Shanks [1] carried out a numerical study of Littlewood's bounds (Littlewood [1]):

$$[(1 + o(1))(\frac{12C}{\pi^2})\log\log|d|]^{-1} < L(1, \chi) < (1 + o(1))2C\log\log|d|$$

where $\log C$ is the Euler constant, $\chi = (d|n)$ the Kronecker symbol, d lies in the range $|d| < 2\times 10^{13}$ and chosen to minimize and maximize $L(1, \chi)$. Shank also showed that the coefficients 12 and 2 in the Littlewood's bounds become $\frac{6}{1-\alpha}$ and $\frac{1}{1-\alpha}$ if the real parts of the zeros of the L-function are constrained to be no greater than α .

Now if χ as a nonreal primitive character mod q , Kanemitsu [1] proved that

$$|L(1, \chi)| < \frac{1}{2}\log q + 3 .$$

4.3.2 Secondly, if χ is a real non-principal character mod q ,

J. Pintz [1] showed that there exists a constant c such that

$$\frac{c}{\sqrt{q}\log q} < |L(1, \chi)| .$$

§4.4. Nonvanishing region

4.4.1 If χ is an odd character (mod q), it is an unsolved problem whether

$$L(s, \chi) > 0 \quad (s > 0) .$$

In S. Chowla [3] the authors proposed a hypothesis which ensures the truth of this inequality. They called the hypothesis J:

there exists a set of primes $p_1, \ldots, p_g$ with the Jacobi symbol

$(\frac{p_m}{q}) = -1 \quad (1 \le m \le g)$

and such that

$$\sum_{j=1}^{w} \chi^*(j) \ge 0 \quad \text{for all } w .$$

Here $$\chi^*(j) = \chi(j)\chi_o(j) ,$$

where χ_o is the principal character mod P and $P = p_1 p_2 \ldots p_g$, i.e.,

$$\chi_o(j) = (\frac{j}{p_1})^2(\frac{j}{p_2})^2 \ldots (\frac{j}{p_g})^2 .$$

Theorem: $L(s, \chi) > 0$ if hypothesis J is true.

4.4.2 Pintz [2]III: For ε such that $0 < \varepsilon < 1/_8$, and $q < q_1(\varepsilon)$ ($q_1(\varepsilon)$ is an effective constant depending on ε), let

$$H(\varepsilon, q) = \{s;\ s = 1 - \tau + it,\ |1 - s| \geq \frac{1}{\log^4 q}\ ,$$

$$0 \leq \tau \leq \frac{1}{4} - \varepsilon\ ,\ |s| \leq q^{(\frac{1}{4} - \frac{\varepsilon}{2})\frac{1}{\rho} - \frac{3}{4}}$$

$$\text{where} \quad \rho = \max(\tau,\ q^{-\varepsilon/4})\}\ .$$

If we assume the inequality

$$h(-q) \leq (\log q)^{3/4}$$

where $h(-q)$ is the class number of the imaginary quatratic field belonging to the fundamental discriminant $-q < 0$, then neither $L(s, \chi)$, $(\chi(n)) = (-q/n)$ nor $\zeta(s)$ has a zero in $H(\varepsilon, q)$.

4.4.3 Let $\psi(x) = \Sigma_{n \leq x} \Lambda(n)$, $\Delta(x) = \psi(x) - x$, where $\Lambda(n)$ is the Mangoldt function. P. Turan [1] proved that if γ_1 is the supremum of the numbers γ for which $\Delta(x) = O(xe^{-c_1 \log^{\gamma} x})$, and γ_2 is the infimum of the numbers γ' for which the Riemann zeta function $\zeta(s) \neq 0$ in the region $\sigma > 1 - c_2/\log^{\gamma'}|t|$, $|t| \geq c_3$; c_1, c_2, c_3 denoting numerical constants, then $\gamma_1 = 1/_{(1+\gamma_2)}$. In K. Wiertelak [1], the author proved similar results for the Dirichlet L-functions. Write $\psi(x, q, \ell) = \Sigma_{n \equiv \ell (\mathrm{mod}\ q) n \leq x} \Lambda(n)$, where $q \geq 1$, $0 < \ell \leq q$, $(\ell, q) = 1$, and let χ be a character mod q. Let χ_0 denote the principal character and χ_1 the exceptional character mod q. Further, let $\Delta(x, q, \ell) = \psi(x, q, \ell) - x/\varphi(q) + (\chi_1(\ell)/\varphi(q)) \cdot x^{\beta_1}/\beta_1$; β_1 (if it exists) denoting the exceptional real zero of $L(s, \chi_1)$. The following results are proved.

Theorem: If $0 < \gamma \leq 1$, $w(q)$ is a positive function, and $\Delta(x, q, \ell) < c_1(x/\varphi(q)) \cdot \exp(-c_2(\log x/\max(\log q, \log^{\gamma/1+\gamma} x)))$ (for $x \geq \omega(q)$ and any ℓ), then $\Pi_{\chi \bmod q} L(s, \chi) \neq 0$ in the region

$\sigma > 1 - (1/30)(c_2/2)^{1+\gamma}/\max\{\max(1, (c_2/2)^{1+\gamma})\log q, \log^{\gamma}|t|\}$,

$|t| \geq \max(c_3, q^{1/9}, q^{-1}\exp((\log \omega(q))/\log q))$, $q \neq 1$; $|t| \geq \max(c_3, q^{1/9})$,

$q = 1$, where c_2 depends at most on c_1, c_2 and γ .

Theorem: If γ_1 is the supremum of the numbers γ for which

$|\Delta(x, q, \ell)| < c_1(x/\psi(q)\exp\{-c_1(\log x/\max(\log q, \log^{1-\gamma}x))\}$ $(x \geq \omega(q))$,

where $\omega(q)$ is any function satisfying $\exp \log^2 q \leq \omega(q) \leq \exp(A \log^2 q)$

with the constant $A \geq 1$, and if γ_2 is the infimum of γ' for which

$\Pi_{\chi \bmod q} L(s, \chi) \neq 0$, $s \neq \beta_1$ in the region $\sigma > 1 - c_3/\max\{\log q \log^{\gamma'}(|t| + 3\}$,

then $\gamma_1 = 1/(1+\sigma_2)$.

In Y. Motohashi [3], the author gave a new proof of I.M. Vinogradov's bound for the zero-free region for the Riemann zeta function.

V. ZEROS OF L-FUNCTIONS

§5.0. There are two kinds of results concerning the zeros of L-functions. One kind deals with the zeros lying on the real axis which is intimately related to the question of nonvanishing regions of L-functions (and so perhaps should be included in the previous chapter). The other kind of results deals with the density of zeros in a rectangle in the critical strip and is related to the famous Riemann hypothesis which we must recall.

Riemann hypothesis (RH): all the nontrivial zeros of the ζ function lie on the line $\mathrm{Re}\ s = \frac{1}{2}$.

Generalized Riemann hypothesis (GRH): all the nontrivial zeros of $L(s, \chi)$ lie on the line $\mathrm{Re}\ s = \frac{1}{2}$.

Weak generalized Riemann hypothesis ($GR_\delta H$): for some δ such that $\frac{1}{2} \le \delta < 1$, the real parts of all the nontrivial zeros of $L(s, \chi)$ are not bigger than δ.

Yu. V. Linnik is certainly the first person to recognize the power of zero-density estimates in proving arithmetic theorems: In Linnik [5] he proved Vinogradov's theorem that every sufficiently large odd integer in the sum of three odd primes; in Linnik [2] he showed that the least prime $p(q, a)$ in the arithmetic progression $nq + a$ $((a, q) = 1)$ satisfies the inequality:

$$p(q, a) \ll q^L$$

(L is called the Linnik's constant, cf. Jutila [7]). Ingham [4] also used zero density estimates to study the difference between two consecutive primes (recently M.N. Huxley [5] showed that a slight modification of the agrument in Montgomery [4] allows us to conclude that there is a prime number between x and $x^{\frac{7}{12}+\varepsilon}$) .

Zero-density estimates for $\sum_{\chi} N(\alpha, T, \chi)$ and $\sum_{q\le Q} \sum_{\chi}^{*} N(\alpha, T, \chi)$ were essentially first obtained by Ingham ([3],[4]). Significant subsequent improvements were given by, Bombieri, Montgomery, Huxley and Jutila.

§5.1. Real zeros

Recent results on real zeros were those re-proved by Pintz by elementary methods..

5.1.1 A. Page [1] proved in 1934 the following theorem: If an L-function belonging to a real primitive character modulo q has a real zero $1 - \delta$, then

$$\delta >> \frac{1}{\sqrt{q}\,\log^2 q} .$$

The real zeros $1 - \delta$ with $\delta \le 1/_{\log q}$ we shall call Siegel-zeros. As the ineffectiveness of the Siegel-zeros causes a lot of difficulties in analytical number theory, it is natural to try to improve the above inequality in an effective way. In J. Pintz [2]II, the author gave some effective results.

Theorem 5.1. For the greatest real zero $1 - \delta$ of an L-function belonging to a real primitive character modulo q , the inequality

$$\delta \ge \frac{12 - o(1)}{\pi\sqrt{q}}$$

holds.

As an L-function belonging to an imprimitive character modulo q has the same zero as an other L-function belonging to a primitive character with a modulo $q^* < q$, theorem 5.1 remains true for imprimitive character too.

5.1.2 In Pintz [2]II & III, it is proved that

Theorem 5.2. If the inequality

$$h(-q) \le \frac{\log q}{2 \log \log q}$$

holds, where $h(-q)$ is the class number of the imaginary quadratic field belonging to the fundamental discriminant $-q < 0$, then (1) for the greatest real zero $1 - \delta$ of $L(s, \chi)$, where $\chi(n) = (\frac{-q}{n})$, the relation

$$\delta = \frac{L(1, \chi)}{\prod_{p|q} (1 + \frac{1}{p})\frac{\pi^2}{6}} (1 + o(1)) = \frac{6h(-q)}{\prod_{p|q} (1 + \frac{1}{p})\pi\sqrt{q}} (1 + o(1))$$

holds; (2) $L(s, \chi)$ has a single simple real zero $1 - \delta$ in

$$H = \{s; |1 - s| \le \frac{1}{\log^4 q}\} ,$$

and for which one has

$$\delta = \frac{L(1, \chi)}{\prod_{p|q} (1 + \frac{1}{p})\frac{\pi^2}{6}} (1 + O(\frac{1}{\log q})) = \frac{6h(-q)(1 + O(\frac{1}{\log q}))}{\prod_{p|q} (1 + \frac{1}{p}) \pi \sqrt{q}}$$

5.1.3 M. Deuring [1] and H. Heilbronn [1] discovered that the non-trivial zeros of $\zeta(s)$ and $L(s, \chi)$ (where χ is an arbitrary real or complex character), have influence on the real zeros of other real L-functions. In J. Pintz [2]IV. the author studied the influence of the nontrivial zeros on the exceptional zeros. He obtained the following theorem.

Theorem 5.3. If an L-function belonging to a non-principal character χ_k mod k has an $s_o = 1 - \gamma + it$ zero with $\gamma < 0.05$, and an other L-function belonging to the real non-principal character χ_q (for which $\chi_k \chi_q$ is also non-principal) mod q has an $1 - \delta$ real exceptional zero, then the inequality

$$\delta > \frac{1}{140\, u^{6\gamma} \log^5 u}$$

holds, where $u = k|s_o|q$.

5.1.4 E. Landau [1] proved in 1918 that if the L-functions belonging to real primitive character $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ $(\chi_1 \neq \chi_2)$ respectively, have $1 - \delta_1$ and $1 - \delta_2$ real zeros, respectively, then

$$\max(\delta_1, \delta_2) > \frac{c}{\log q_1 q_2}$$

where c is an absolute constant.

A. Page [1] proved the above inequality for the case $\chi_1 = \chi_2$, i.e. he showed that an L-function belonging to a real non-principal character $\chi \pmod q$ has at most one, simple zero in the interval $[1 - \frac{c''}{\log q}, 1]$ where c'' is an absolute constant.

In J. Pintz [2]V the author proved Page's theorem and Landau's theorem in a relatively simple way, using only elements of the real analysis.

§5.2. Nonreal zeros

5.2.1 If χ is a primitive character, $A(t, t_1)$ denote the number of zero of $L(\frac{1}{2} + iu, \chi)$. V.G. Zuravlev [1] showed that

$$A(t, t_1) > c(\psi(k)/k)(t - t_1)\log t \quad \text{for} \quad t - t_1 \geq k^2 t^{\alpha} ,$$

$\frac{1}{2} < \alpha \leq 1$, $t > c_1$, $t_1 \geq 0$ where $c > 0$ and $c_1 > 0$ are constants that depends only on α .

5.2.2 S.S. Parmankulov [1] proved that $L(\frac{1}{2} + it, \chi)$ has more than $AT \log T$ zeros in $(0, T)$ for all $T > T_o = D^{1+\varepsilon}$.

5.2.3 P.J. Weinberger [1] presented a method of finding zeros of $L(s, \chi)$ with $s = \frac{1}{2} + it$ and t small.

5.2.4 Let χ be a Dirichlet character modulo k and $N_o(T, \chi)$ the number of zeros of $L(\frac{1}{2} + it, \chi)$ with $0 < t < T$. V.G. Zurralev [2] proved that $N_o(T + u, \chi) - N_o(T, \chi) > Au \log T$ with $A = \frac{\theta}{5 + 2\pi}$, θ any constant in $(0, \frac{1}{4})$, $u > T^{\frac{1}{2}+2\theta}$ and $T > C(\theta, k)$.

5.2.5 In S.M. Voronin [1], the author proved the existence of a sequence $x_r \to \infty$ such that the function $\sum_{n \leq x_r} n^{-s}$ has zeros in the half-plane $\mathrm{Re}\, s > 1$. Also he proved that if $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, then there exists $x_o = x_o(\sigma_1, \sigma_2)$ such that for $x > x_o$, the function $\sum_{n \leq x} n^{-s}$ has infinitely many zeros in the strip $\sigma_1 < \mathrm{Re}\, s < \sigma_2$. Analogous results are stated for partial sums of L-functions.

5.2.6 A complex number ρ is called a coincident zero of $L(s, \chi_1)$ and $L(s, \chi_2)$ if $L(\rho, \chi_1) = L(\rho, \chi_2) = 0$ with the same multiplicities. A. Fujii [2] showed that for two distinct primitive characters of the same modulus, a positive proportion of the zeros are non-coincident.

5.2.7 A. Fujii [7] take care on the uniformity of the distribution of the zeros of the Riemann zeta function $\zeta(s)$ and of Dirichlet L-functions $L(s, \chi)$. Let

$A = \{a_n; n = 1, 2, 3, \dots\}$ be a sequence of non-negative real numbers and $B = \{a_n; n = 1, 2, 3, \dots\}$ be a sequence of increasing non-negative real numbers.

For each $\alpha \in (0, 1]$, let $A(x, \alpha)$ be the number of $a_n < x$ such that

$$b_{m-1} \leq a_n < b_m \quad \text{and} \quad \frac{a_n - b_{m-1}}{b_m - b_{m-1}} < \alpha \quad \text{for some } m .$$

Put $A(x) = A(x, 1)$.

Definition. A is uniformly distributed modulo B if for each $\alpha \in (0, 1)$, $\lim_{x \to \infty} \frac{A(x, \alpha)}{A(x)} = \alpha$. Let $\gamma_n(\chi)$ be the n-th non-negative ordinate of the zeros of $L(s, \chi)$ in the critical strip.

Theorem. For primitive characters χ_1 and χ_2, $A(\chi_1)$ is uniformly distributed modulo $B(\chi_2; a, b)$ for any $a > 0$ and $b > 0$, where χ_1 may be equal to χ_2, where

$$A(\chi_1) = \{\gamma_n(\chi_1); n = 1, 2, 3, \dots\}$$

$$B(\chi_2; a, b) = \{b_n; b_n = 0 \text{ when } 0 \leq \gamma_n(\chi_2) \leq 1$$
$$\text{and } b_n = b\gamma_n(\chi_2)(\log \gamma_n(\chi_2))^a$$
$$\text{when } \gamma_n(\chi_2) > 1\} .$$

The theorem is true for $A(\chi_1) = \{\gamma_n(\chi_1); n \geq N_o\}$ and $B(\chi_2; b, d) = \{b_n; b_n = b\gamma_n(\chi_2)\log_\alpha \gamma_n(\chi_2) \text{ for } n \geq N_o\}$.

§5.3. Density of zeros

We shall use the notation of §1.4.

5.3.1 If $T^{\alpha} \le H \le T$, where $\frac{1}{2} + \frac{\max(8,\ 4\log q)}{\log T} \le \alpha \le 1$, $T \ge \max\{e^{16}, q^{8}\}$, T. Fryoka [1] showed that for any primitive character $\chi \pmod q$, one has

$$N(\alpha, T + H, \chi) - N(\alpha, T, \chi) = O(qH(^{H}/_{\sqrt{T}})^{(1-2\alpha)/4}\log T)$$

where q is the conductor of χ and the constant implied by the O depends only on α , this is a generalization of A. Selberg [1].

In T. Fryska [2], the author gave an improvement:

$$N(\alpha, T, \chi) = O(q^{1-\alpha}\log^{2}(q+1)T^{1+(1+2\alpha)/8}\log(2qT))$$

uniformly for $\frac{1}{2} \le \alpha \le 1$, where χ is of conductor q . If $q \le T^{\frac{1}{4}-\varepsilon}$, then $N(\alpha, T, \chi) = O(T^{1+(1-2\alpha)/8}\log T)$. T. Hilano [1], [2] proved an analogue to the famous theorem of N. Levinson [5]. Let χ be a primitive character (mod q) . For each $\varepsilon > 0$, assume that $\log q \le (\log T)^{1-\varepsilon}$. Set $L = \log(qT/_{2\pi})$ and $u = T/_{(qL^{4})}$. Let $N_{o}(T, \chi)$ and $N(T, \chi)$ have their usual meanings. Then the author showed that $N_{o}(T + u, \chi) - N_{o}(T, \chi) > \frac{1}{3}(N(T + u, \chi) - N(T, \chi))$, which implies that at least $\frac{1}{3}$ of the nontrivial zeros of the Dirichlet L-function $L(s, \chi)$ lie on the critical line.

5.3.2 The best known general upper estimate of $\sum_{\chi} N(\alpha, T, \chi)$ is the result (1.19) of Montgomery which is obtained by using certain theorems on Dirichlet polynomials involving Dirichlet characters, and a mean-value estimate for $|L(\frac{1}{2} + it, \chi)|^{4}$. In 1972 M. Jutila [6] showed that the method of Montgomery could be refined to yield the following

Theorem 5.4. For any fixed $\varepsilon > 0$ there exist (calculable) numbers $C = C(\varepsilon)$, $B = B(\varepsilon)$ such that uniformly for $\frac{1}{2} \le \alpha \le 1$, $T \ge 2$, $Q \ge 1$.

$$N(\alpha, T, Q) \leq C(QT)^{(\omega(\alpha)+\varepsilon)(1-\alpha)}(\log QT)^B$$

with

$$\omega(\alpha) = \frac{3}{2-\alpha} \quad \text{for} \quad \frac{1}{2} \leq \alpha \leq \frac{\sqrt{17}-1}{4} = 0.78077\ldots$$

$$\omega(\alpha) = \frac{6\alpha-3}{6\alpha-4} \quad \text{for} \quad \frac{\sqrt{17}-1}{4} \leq \alpha \leq \frac{5}{6}$$

$$\omega(\alpha) = 2 \quad \text{for} \quad \frac{5}{6} \leq \alpha \leq 1 .$$

Consequently, in any case,

$$N(\alpha, T, Q) \leq C(QT)^{(\omega_o+\varepsilon)(1-\alpha)}(\log QT)^B$$

with $\qquad \omega_o = \frac{3}{16}(9+\sqrt{17}) = 2.46058\ldots$

5.3.3 In 1975, Y.Motohashi [2] proved that if $\alpha \geq 4/_5$, $\sum_{\chi \bmod q} N(\alpha, T, \chi) \ll_\varepsilon (q^2T^3)^{(1+\varepsilon)(1-\alpha)}$. In Y. Motohashi [5], the author showed that for α close to 1, the right hand side may be replaced by $(q^{9/8}T^6)^{(1+\varepsilon)(1-\alpha)}$.

5.3.4 In 1977, M. Jutila [1] showed that if ε is any fixed positive number and k any fixed positive integer. Then for $3/_4 \leq \alpha \leq 1$, $T \geq 2$, we have

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi (\bmod q)} N(\alpha, T, \chi) \ll_{\varepsilon,k} (Q^2T)^{A_1(\alpha)(1-\alpha)+\varepsilon},$$

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi (\bmod q)} N(\alpha, T, \chi) \ll_{\varepsilon,k} (Q^2T^2)^{A_2(\alpha)(1-\alpha)+\varepsilon},$$

where $\quad A_1(\alpha) = \max\left(2, \frac{3}{(8k-3)\alpha+3-6k}, \frac{3k}{(4k-3)\alpha+3-2k}\right)$

$$A_2(\alpha) = \max\left(2, \frac{5}{(16k-5)\alpha+5-12k}, \frac{5k}{(8k-5)\alpha+5-4k}\right) .$$

(see 5.3.11.)

In particular, choosing $k = 2$ we have $A_1(\alpha) = 2$ for $\alpha \geq 21/_{26}$; choosing $k = 4$ we have $A_2(\alpha) = 2$ for $\alpha \geq 7/_9$.

In 1979, D.R. Heath-Brown [5] improved the above results as follows:

$$\sum_{q\leq Q} \sum^*_{\chi(\text{mod } q)} N(\alpha, T, \chi) \ll_{\varepsilon,k} (Q^2T^\beta)^{A(\alpha)(1-\alpha) + \varepsilon}$$

where (i) $\beta = 1$, $A(\alpha) = 2$ and $11/_{14} \leq \alpha \leq 1$.

(ii) $\beta = 2$, $A(\alpha) = 2$ and $129/_{167} \leq \alpha \leq 1$.

Note that (a) $21/_{26} = 0.807 \ldots$ and $11/_{14} = 0.785 \ldots$

(b) $7/_9 = 0.7777 \ldots$ and $129/_{167} = 0.7724 \ldots$

In the proof of (i), the author applied the zero detection method, which are given in Huxley [4], to (1.3) of Jutila [1].

To prove (ii), the author modified the method of Jutila [1] -- he made use of the lemmas of Jutila [1] successively to get a better estimate of the integral function

$$\sum_{r,s=1}^{k} \left| \sum_{N<n\leq 2N} \bar{\chi}_r \chi_s(n) n^{-\frac{1}{2} + i(\gamma_r - \gamma_s + \tau)} \right| .$$

5.3.5 In 1979, D.R. Heath-Brown [1] showed that

$$\sum_{\chi \text{ mod } q} N(\alpha, T, \chi) \ll (qT)^{2 - 2\alpha + \varepsilon} , \quad \frac{15}{19} \leq \alpha \leq 1 .$$

5.3.6 H.L. Montgomery [4] dealt with the problem of obtaining bounds for the quantity $\sum_{q\leq Q} \sum^*_{\chi} N(\alpha, T, \chi)$ where $\alpha \geq \frac{1}{2}$, $T \geq 1$.

When T is large and $Q = 1$, A.E. Ingham [3] gave the result:

$$N(\alpha, T) \ll T^{\frac{3(1-\alpha)}{2-\alpha}} (\log T)^5 , \tag{5.1}$$

and when $T \leq Q$, E. Bombieri [4] showed that

$$\sum_{q \leq Q} \sum_{\chi}^{*} N(\alpha, T, \chi) << TQ^{\frac{8(1-\alpha)}{3-2\alpha}} (\log Q)^{10} . \qquad (5.2)$$

Both (5.1) and (5.2) are proved by the use of mean values theorems; the large sieve is also used in the case of (2.2). The lack of a bound for $\sum_{q \leq Q} \sum_{\chi}^{*} N(\alpha, T, \chi)$ which is good with respect to both Q and T may be traced to the absence of a sufficiently strong mean value theorem; using the recent results concerning mean and large values of Dirichlet polynomials (H.L. Montgomery [3]).

H.L. Montgomery [4] gave the following theorem:

Theorem 5.5. For $Q \geq 1$, $T \geq 2$, and $\frac{1}{2} \leq \alpha \leq \frac{4}{5}$.

$$\sum_{q \leq Q} \sum_{\chi}^{*} N(\alpha, T, \chi) << (Q^2 T)^{\frac{3(1-\alpha)}{2-\alpha}} (\log QT)^{13} ,$$

and for $\frac{4}{5} \leq \alpha \leq 1$

$$\sum_{q \leq Q} \sum_{\chi}^{*} N(\alpha, T, \chi) << (Q^2 T)^{\frac{2(1-\alpha)}{\alpha}} (\log QT)^{13} .$$

In the case $Q = 1$, Theorem 5.5 gave

$$N(\alpha, T) << T^{\frac{5}{2}(1-\alpha)} (\log T)^{13} . \qquad (5.3)$$

uniformly for $\frac{1}{2} \leq \alpha \leq 1$.

5.3.7 In 1970, Gallagher [1] gave a common basis for the device of Fogel [1] (Turan's power sum method) and Montgomery [4]. He proved that

$$\sum_{q \leq T} \sum_{\chi}^{*} N(\alpha, T, \chi) << T^{c(1-\alpha)} \quad (T \geq 1) \qquad (5.4)$$

The methods used in the proof of (5.4) are a general mean value estimate for exponential sums, a large sieve estimate, due to Bombieri and Davenport, for character sums with prime argument, and an application of Turan's power sum lemma.

5.3.8 For a fundamental discriminant d, let $\chi_d(n) = (d/_n)$ and define $N_r^*(\sigma, T, X) = \Sigma_{|d|\leq X} N(\sigma, T, \chi_d)$ and $N_r'(\sigma, T, X) = \Sigma_{|d|\leq X}(X|d|^{-1})^{\frac{1}{2}} N(\sigma, T, \chi_d)$. M. Jutila [5] stated that, uniformly for $\frac{1}{2} \leq \sigma \leq 1$, $T \geq 2$ and $X \geq 3$, $N_r'(\sigma, T, X) \ll X^{(7-6\sigma)/(6-4\sigma)} T^2 \log^{68}(XT)$. One corollary is a similar estimate for $N_r^*(\sigma, T, X)$.

5.3.9 Motohashi [2] proved that if $\alpha \geq \frac{4}{5}$ then

$$\Sigma_{q\leq Q} \Sigma^*_{\bmod q} N(\alpha, T, \chi) \ll_\varepsilon (Q^4 T^3)^{(1+\varepsilon)(1-\alpha)}$$

and for α close to 1 he [5] showed that the right hand side may be replaced by $(Q^{9/4} T^6)^{(1+\varepsilon)(1-\alpha)}$.

5.3.10 In M.N. Huxley [3] the author gave three theorems. The idea in proving the first theorem is to consider the large values of the product of a Dirichlet polynomial with its "conjugate", this can be done successfully since the product is of "good" length. As a result, the author generalized in this theorem the key estimate of his previous paper [Invent Math. 15(1972), 164-170, MR 45#1856] to Dirichlet polynomials containing a variable Dirichlet character, and hence obtained the qT- and Q^2T-analogues of the zero-density estimate $N(\alpha, T) \ll T^{12(1-\alpha)/5 + \varepsilon}$. The second theorem is a combinatoric refinement of the first theorem, and the third theorem contains the estimates needed in the zero-density problems.

In 1976 M.N. Huxley [4], the author used the Dirichlet polynomial technique to prove a zero-density theorem for L-functions:

$$\sum_{q\le Q} {}^{q}/_{\psi(q)} \sum_{\chi \bmod q}^{*} N(\alpha, T, \chi) \ll_{\varepsilon} (QT)^{A(\alpha)(1-\alpha)+\varepsilon},$$

where
$$A(\alpha) = \begin{cases} 10/_{(3-\alpha)} & \text{for } \frac{1}{2} \le \alpha \le \frac{3}{4} \; ; \\ 10/_{(7\alpha-3)} & \text{for } \frac{3}{4} \le \alpha \le \frac{11}{14} \; ; \\ 4 & \text{for } \frac{11}{14} \le \alpha < 1 \; . \end{cases}$$

For all α it is possible to take $A(\alpha) = 40/_9$.

5.3.11 Let a Dirichlet polynomial of the type $f(s, \chi) = \sum_{N+1}^{2N} a_n \chi(n) n^{-s}$ with variable Dirichlet character χ and variable complex number s be given. Suppose we are given a set of pairs (s_r, χ_r) , $r = 1, \ldots, R$, where the points $s_r = \sigma_r + it_r$ satisfy $\sigma_r \ge 0$, $|t_r - t_s| \le T$, and for $r \neq s$ either $\chi_r \neq \chi_s$ or $|t_r - t_s| \ge 1$. Further, suppose that $|f(s_r, \chi_r)| \ge V > 0$ for all r . In 1977 M. Jutila [1] estimated the number R in three cases (i) all χ_r belong to the same modulus q ; (ii) all χ_r are primitive characters of conductor at most Q ; (iii) χ_r is the principal character χ_o (mod 1) for all r , and hence gave the following new zero-density estimates:

Theorem 5.6. Let ε be any fixed positive number and k be any fixed positive integer. Then for $\frac{3}{4} \le \alpha \le 1$, $T \ge 2$, we have

$$\sum_{q\le Q} \sum_{\chi}^{*} N(\alpha, T, \chi) \ll_{\varepsilon,k} (Q^2T)^{A_1(\alpha)(1-\alpha)+\varepsilon}, \tag{5.5}$$

$$\sum_{q\le Q} \sum_{\chi}^{*} N(\alpha, T, \chi) \ll_{\varepsilon,k} (Q^2T^2)^{A_2(\alpha)(1-\alpha)+\varepsilon}, \tag{5.6}$$

$$N(\alpha, T) \ll_{\varepsilon,k} T^{A_3(\alpha)(1-\alpha)+\varepsilon}$$

where

$$A_1(\alpha) = \max\left(2, \frac{3}{(8k-3)\alpha + 3 - 6k}, \frac{3k}{(4k-3)\alpha + 3 - 2k}\right)$$

$$A_2(\alpha) = \max(2, \frac{5}{(16k-5)\alpha + 5 - 12k}, \frac{5k}{(8k-5)\alpha + 5 - 4k})$$

$$A_3(\alpha) = \max(2, \frac{3}{(8k-3)\alpha + 3 - 6k}, \frac{3k}{(3k-2)\alpha + 2 - k}) .$$

5.3.12 K. Ramachandra gave some results on the zeros of a class of generalized Dirichlet series: A sequence of complex numbers $\{a_n\}$ is said to be "good" if $a_1 \neq 0$ and if there exist constants. $(0 <)c < C$ and a continuous, monotonic function $G(x)$, of slow variation, such that $c\, G(x) < x^{-1} \Sigma_{n\le x} |a_n|^2 < C\, G(x)$ for $x \ge 1$. If also $0 < \lambda_1 < \lambda_2 < \dots$ with $0 < a \le \lambda_{n-1} - \lambda_n \le b$, let $F(s) = \Sigma_{n=1}^{\infty} a_n \lambda_n^{-s}$, with convergence of that Dirichlet series in some non-empty half-plane. The main result of K. Ramachandra [2] was the following theorem: Let $\{a_n\}$ be a good sequence, let the previous Dirichlet series represent a function of finite order, continuable as a meromorphic function, with a finite number of poles on the half-plane $\alpha \ge \sigma$ (with $\sigma < \frac{1}{2}$) ; let $N(\alpha, T)$ denote the number of zeros of $F(s)$ with real part $\ge \alpha$ and imaginary part in $[-T, T]$; then, for fixed α in $\sigma < \alpha < \frac{1}{2}$ and $T \to \infty$,

$\log N(\alpha, T) = \log T + O_{\ell}((\log T)^{(2\lambda-1)/2(\lambda-\alpha) + \varepsilon})$, for arbitrary $\varepsilon > 0$ and real λ satisfying $|a_1\lambda_1| > \Sigma_{n=2}^{\infty} |a_n \lambda_n^{-\lambda}|$. Many of the conditions indicated can be weakened or otherwise modified. The interest of this theorem is one obtains $\log N(\alpha, T) \sim \log T$, just as for Riemann's zeta function. The author also proved that $F(s)$ can be continued analytically in $\alpha > \sigma$, $t \ge t_o > 0$, while satisfying these $|F(s)| < t^A$ for some $A > 0$; let λ be the abscissa of absolute convergence of the series representing $F(s)$ and let c be a constant such that $(2\lambda - 1)/2(\lambda - \sigma) < c \le 1$, then, if also $\Sigma_{n\le x} |a_n|^2 > x^{1-\varepsilon}$ holds and $T \ge T_o(\sigma, c, A)$, and if we define $V = \exp(\log^c T)$, there is at least one zero in $\alpha \ge \sigma$; $T \le t \le T + V$. In K. Ramachandra [3], the author imposed the additional condition $|a_n| \le A_\varepsilon n^\varepsilon$ (for every fixed $\varepsilon > 0$) and considered only the particular case $\lambda_n = n$. Under these stronger conditions, the author strengthened the statement of the previous theorem "at least one zero" to "at least $V \exp(-\log^\delta T)$ zeros" for any $\delta > 0$ and $T \ge T_o(\sigma, c, A, \delta)$.

§5.4. Zeros of $\zeta(s)$

We include in this section recent results on the zeros of the Riemann zeta function. Apart from the long efforts of Fujii on the distribution of the ordinates of zeros, most of the results are related to the Riemann hypothesis in one way or another. As they do not fall into a clear cut pattern, it is perhaps best just to list them.

5.4.1 In H.J. Besenfelder, G. Palm [1], the authors proved

$$(*) \quad \lim_{T\to\infty} \sum_{|\mathrm{Im}(\rho)|<T} \exp[y(\rho - \tfrac{1}{2} - it)^2] = 2\exp[y(\tfrac{1}{4} - t^2)]\cos(ty)$$
$$- (\log \pi + \gamma)/2\sqrt{\pi y} - (1/\sqrt{\pi y})\sum_{m=2}^{\infty} \Lambda(m)m^{-\frac{1}{2}}\exp[-\log^2 m/4y]\cos(t \log m)$$
$$+ (1/\sqrt{\pi y})\int_o^\infty (1 - \exp(-x^2/_{4y} + 3x/_2)\cos(tx)/(\exp[2x] - 1)dx .$$

where the sum on the left hand side is over the non-trivial zeros of the zeta function and $\Lambda(m)$ is the von Mangoldt function. The method of proof involves estimating both sides of (*) under the assumption of the truth and falsity of the Riemann hypothesis. This leads to the following theorem: The following statements (a) to (f), are equivalent to the Riemann hypothesis, if t is an arbitrary real number, then both sides of (*) are

(a) positive for all $y > 0$;

(b) bounded as $y \to \infty$;

(c) convergent as $y \to \infty$, to 0, if t is not the abcissa of a zero of $\zeta(s)$, or to v, if t is the abcissa of a zero of multiplicity v.

If $t > \frac{1}{2}$ and $\varepsilon > 0$, then

(d) $(1/\sqrt{\pi y})\Sigma_{m=2}^{\infty}\Lambda(m)m^{-\frac{1}{2}}\exp(-\log^2 m/_{4y})\cos(t \log m) \le \varepsilon$ for sufficiently large y.

If $t > \frac{1}{2}$, then $(1/\sqrt{\pi y})\Sigma_{m=2}^{\infty}\Lambda(m)m^{-\frac{1}{2}}\exp(-\log^2 m/_{4y})\cos(t \log m)$ is

(e) convergent to 0 or $-v$, as $y \to \infty$.

The statement of theorem for L-functions is similar and used a formula corresponding to (*).

5.4.2 R. Spira [2] showed that the Riemann hypothesis is equivalent to the condition that $|\zeta(s)|$ increases as Re s moves left from $\frac{1}{2}$ for Im s sufficiently large.

5.4.3 V.G. Sprindzuk [1] gave a necessary and sufficient condition for truth of the extend Riemann hypothesis in terms of the imaginary parts of the zeta-zeros under the assumption of the Riemann hypothesis for $\zeta(s)$. The author also showed that if the Riemann hypothesis for $\zeta(s)$ is true, then "very likely" the Riemann or at least a quasi-Riemann hypothesis is true for $L(s, \chi)$.

5.4.4 In Y. Tomonaga; S. Hiyama [1], the authors were concerned with the summatory for $M(x) = \Sigma_{n \le x}\mu(n)$ of the Mobius function, they stated that $M(x) = O(x^{\frac{1}{2}+\varepsilon})$ for each $\varepsilon > 0$ is a necessary and sufficient condition for the Riemann hypothesis. The authors took attention to the partial sums

$S_N(s) = \Sigma_{n=1}^{N} \mu(n)/_{n^s}$ for $1/_{\zeta(s)}$ in Y. Tomonaga; S. Hiyama [2], since $\lim_{N\to\infty} S_N(s) = 1/_{\zeta(s)}$ for each $s > \frac{1}{2}$ is also an equivalent statement for the Riemann hypothesis.

5.4.5 A.E. Ingham [2] introduced a series sequence summability method which was called (I)-summability by G.H. Hardy [1], and proved that $(C, -\delta) \to (I) \to (C, \delta)$ for every $\delta > 0$, where (C, k) denotes the Cesáro means of other k. However, Ingham published only the implication $(I) \to (C, 1)$, remarking that the result that $(I) \to (C, \delta)$ for every $\delta > 0$ requires the use of (**) $\Sigma_{n \leq x} \mu(n)/_n = 0((\log x)^{-k})$ for every $k > 0$. W.B. Pennington [1] gave another unpublished proof of the result that $(I) \to (C, \delta)$ for every $\delta > 0$ where (**) also required. Now, in S.L. Segel [1], the author defined a new class of summability methods similar to Pennington's which includes the Riesz means R_n^{δ}, $0 < \delta < 1$ and which relates connections between (I)-summability and members of this class to hypothesis about the Riemann zeta function. In the appendix the author gave a very simple proof of the result that $(I) \to (C, \delta)$ using only (**) for some $k > 1$.

5.4.6 R. Spira [2] showed that the Riemann hypothesis implied that $\zeta'(s)$ has no zeros in the open left half of the critical strip. Moreover, there is exactly one real zero in each open interval $(-1 - 2n, 1 - 2n)$ for $n = 1, 2, \ldots$.

We now turn to the Riemann zeta function. In R. Spira [4], the author proved that the Dirichlet series which represents $(-1)^k(1 - 2^{1-s})^{k+1}\zeta^{(k)}(s)$ has convergence abscissa zero, where $\zeta^{(k)}$ is the k-th derviative of the Riemann zeta function. This implies that $\zeta'(s)$ has infinitely many zeros in the critical strip.

5.4.7 N. Levinson [12] proved several interesting theorems on the zeros of derviatives $\zeta^{(k)}(s)$ of the Riemann zeta function. Most of the theorems clearly reflect the influence of the zeros of $\zeta(s)$; some give interesting contrasts. We cite just a few of the results. Let $R = \{s = \sigma + it : 0 < \sigma < \frac{1}{2}, 0 < t < T\}$. The authors showed that the number of zeros of $\zeta(s)$ in R differs from the number of zeros of $\zeta'(s)$ there by $O(\log T)$. The authors next prove a quantitative result that implies that most of the zeros of $\zeta^{(k)}(s)$ are clustered about the line $\sigma = \frac{1}{2}$. Let $\rho^{(k)} = \beta^{(k)} + i\gamma^{(k)}$ denote the non-real zeros of $\zeta^{(k)}(s)$, where $k \geq 1$. Then for $0 < u < T$, $2\pi \sum_{T \leq \gamma^{(k)} \leq T+u} (\beta^{(k)} - \frac{1}{2}) \sim k\, u \log \log(T/2\pi)$, which is a weaker form of the result actually obtained. It is also shown that the Riemann hypothesis implies that $\zeta^{(k)}(s)$ has at most a finite number of non-real zeros in the half-plane $\sigma < \frac{1}{2}$.

5.4.8 Let P^n be the subset of $\mathbb{R}^n$ defined by $1 \leq x_1 < x_2 < \ldots < x_n$, for each positive integer n , C. Ryavec [1] constructed a functional ϕ_n , such that if $\rho = \beta + i\gamma$, $0 < \beta < \frac{1}{2}$, is a zero of the Riemann zeta function ζ in the left-half of the critical strip, then $\Phi = \sup_{n\geq 1} \sup_{x \in P^n} \phi_n(x) \leq |\rho|^2/(1 - 2\beta)$. Hence, using the functional equation for ζ , $\Phi = \infty$ implies the Riemann hypothesis.

5.4.9 Let $\gamma_1 \leq \gamma_2 \leq \ldots$ be the sequence of imaginary parts of the zeros of the Riemann zeta function in the upper half of the critical strip, $t = \frac{\log z}{2\pi}$ with an integer $z \geq 2$ and let D_T be the discrepancy of the initial segment of the sequence $t\gamma_1, t\gamma_2, \ldots$ comprised of the terms with $0 \leq \gamma_n \leq T$. E. Hlawka [1] (cf. also Criso [1]) showed that $D_T = o(\frac{\log z}{\log \log T})$. If the R.H. is assumed, this can be improved to $D_T = o(\frac{\log z}{\log T})$.

5.4.10 H.J.J. te Riele [1] presented a table of the imaginary parts of the first 15,000 zeros of $\zeta(s)$ in the critical strip and R.P. Brent [1] showed that the first 75,000,001 zeros of $\zeta(\sigma + it)$ are simple and lie on the critical line $\sigma = \frac{1}{2}$.

5.4.11 Assuming the R.H., J. Mozer [1], [2], [3] described the distribution of the zeros of the Riemann zeta function.

In J. Mozer [4], the author strengthened the theorem of Hardy and Littlewood on the existence of an odd zero $\rho = \frac{1}{2} + it$ in $|t - T| < T^{\gamma+\varepsilon}$. He obtained the new exponent $\gamma = 0.206 \ldots$ in place of the exponent $\gamma = 0.25$ of Hardy and Littlewood.

If the hypothesis of Lindelöf is true, J. Mozer [5] proved the existence of an odd zero of Riemann zeta function on the half-line in the interval $\Delta T \leq T^{\frac{1}{8}+\varepsilon}$.

In J. Mozer [10], the author gave an estimation from above and from below of the constant $A > \log\log \gamma'(\gamma'' - \gamma')$ relative to the vertical distribution of zeros of the Riemann zeta function, where γ' and γ'' are the imaginary parts of consecutive zeros.

Let $S(t) = \pi^{-1} \arg \zeta(\frac{1}{2} + it)$, for a certain infinite subset of pairs of "paired" zeros of the Riemann zeta function, J. Mozer [8] proved that $S(t) = O(\log t/\log\log\log t)$ if $t \in (\gamma_1, \gamma_2)$, where γ_1, γ_2 are the ordinates of the "paired" zeros of these pairs. In J. Mozer [9], the author obtained several statistical results for $S(t)$ and the distance between adjacent zeros of the Riemann zeta function $\zeta(s)$ based on A. Selberg's mean value theorem for $S^{2k}(t)$.

5.4.12 A. Fujii [1] proved that for every large positive T and for an arbitrary given integer $M \geq 2$, the Riemann zeta function has zeros $\beta_j + i\gamma_j$ $(1 \leq j \leq M)$ such that $|T - \sum_{j=1}^{M} \gamma_j| < A(\log T)^{-1}$ where A is some absolute constant possibly depending on M.

Let γ_n be the n-th ordinate of the zeros of the Riemann zeta function $\zeta(s)$ with $0 < \gamma_n \leq \gamma_{n+1}$. For integral $k \geq 1$, $r \geq 1$ let $S_{r,k}(T) = (NT)^{-1} \sum_{T<\gamma_n<2T} \alpha(\gamma_n, r)^k$ where $\alpha(\gamma_n, r) = (\gamma_{n+r} - \gamma_n)/r$. Also let $N_r(C/\log T, T)$ denote the number of γ_n's in $T < \gamma_n < 2T$ with $\alpha(\gamma_n, r) \geq c/\log T$. A. Fujii [6] obtained the following estimates:

(1) Let $T > T_o$, $1 \leq k << (T \log T)^{2/3}$ and $1 \leq r << k^{3/2}$,

then, for some positive absolute constant A,

$$S_{r,k}(T) << (Ak)^{3k^2/(2k+1)} (\log(3+k))^k r^{-2k^2/(2k+1)} (\log T)^{-k} .$$

(2) For $T > T_o$, $C > C_o$ and $1 \leq r \leq T \log TC^{-1}$,

$$N_r(C/\log T, T) << N(T)\exp(-A(rC)^{2/3}(\log rC)^{-1/3}) .$$

Now let γ_n be the n-th positive ordinate of the zeros of $\zeta(s)$. It is well known that $\gamma_n \sim \dfrac{2\pi n}{\log n}$ as $n \to \infty$ (E.C. Titchmarsh [1] 9.4.4). So the following results of A. Fujii [7] become interesting:

Theorem 2. $A(\zeta)$ is uniformly distributed modulo $B(a, b)$ for any $a > 0$, $b > 0$. where

$$A(\zeta) = \{\gamma_n;\ n = 1, 2, 3, \ldots\}$$

$$B(a, b) = \{b_n;\ b_1 = b_2 = 0 \text{ and } b_n = bn(\log n)^{a-1} \text{ for } n \geq 3\}$$

(For Fujii's definition of 'uniformly distributed' see §5.2.7).

Moreover, the author remarked that Theorem 2 is true for $A(\zeta) = \{\gamma_n ; n \geq N_0\}$ and $B(b, d) = \{b_n ; b_n = \frac{b_n}{\log n} \log_\alpha n$ for $n \geq N_0\}$ and put $L(x, \alpha) = (2\pi)^{-1} \int_\alpha^x \log(t/2\pi)dt$, and let $f_b(m)$ denote the number of ordinates γ of zeros of the Riemann zeta function such that $(m - 1)b < L(\gamma, \alpha) \leq mb$. A. Fujii [5] estimated the mean of $f_b(p)$ over primes in an arithmetic progression, and the author discussed the density of m for which $f_b(m) = k$.

5.4.13 R. Spira [1] and N. Levinson [1] gave some results in the zeros of $\zeta_N(S) = \sum_{n=1}^{N} n^{-s}$.

§5.5. Density of zeros of $\zeta(s)$.

In this section we survey the progress made in the density of zeros of Riemann zeta function in the critical strip. We let

$N_0(T)$ be the number of zeros of $\zeta(\frac{1}{2} + it)$ on

$$0 < t \leq T .$$

and

$N(T)$ be the number of zeros of $\zeta(s)$ in

$$0 \leq \sigma \leq 1 , \quad 0 < t \leq T .$$

An historical account as well as the techniques used to prove the existence of zeros of the Riemann zeta-function, $\zeta(s)$; $s = \sigma + it$, on $\sigma = 1/_2$ appears in E.C. Titchmarsh [1] Ch.X.

5.5.1 The most important work on zero density of $\zeta(s)$ which appeared in early seventies is certainly the series of papers of N. Levinson.

Let $F(s) = \pi^{-s/2}\Gamma(s/_2)\zeta(s)$, G.H. Hardy [2] proved that $F(s)$ has infinitely many zeros on the line $\sigma = 1/_2$. In N. Levinson [2] , the author proved that for fixed $\sigma \neq 1/_2$, the number of solutions of $\mathrm{Re}\, F(\sigma + it) = 0$, $0 \leq t \leq T$ is $\geq (T/(2\pi))\log T + O(T)$.

5.5.2 In 1942, A. Selberg [2] showed that there is an effectively computable positive constant c such that $N_0(T) > cN(T)$ where $N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$. Selberg's proof actually goes further and proves the result in $(T, T + u)$ for suitable u . Thus Selberg was the first to show that a finite proportion c of the zeros of $\zeta(s)$ lies on the critical line.

In N. Levinson [3], he sketched remarkable result on the zeros of Riemann's zeta function $\zeta(s)$. By a method different from Selberg, it is shown that at least $1/_3$ of the zeros are on $\sigma = 1/_2$. In order to prove this result, the author showed that it is sufficient to find the change in the argument of $G(s) = \zeta(s) + \zeta'(s)/[f'(s) + f'(1 - s)]$, where $f(s) = \log(\pi^{-3/2}\Gamma(s/_2))$ on $\sigma = 1/_2$.

In N. Levinson [6], with a better choice of parameters, the author proved that at least 0.3427 of the zeros of the $\zeta(s)$ lie on the critical line.

5.5.3 In N. Levinson [5], the author proved that $N_0(T) > \frac{1}{3}N(T)$. The method depended on the fact that the argument of an appropriate function changes sufficiently rapidly. A device of this kind was used by C.L. Siegel [2] to get the Hardy-Littlewood result $N_0(T) > CT$. We now state the theorem of N. Levinson:

Theorem 5.7. Let $u = \dfrac{T}{(\log \frac{T}{2\pi})^{10}}$. Then

$$N_o(T + u) - N_o(T) > \tfrac{1}{3}[N(T + u) - N(T)] .$$

The basic idea of the proof can be developed quickly although the subsequent details are lengthy. Let $h(s) = \pi^{-s/2}\Gamma(s/_2)$. Then $h(s)\zeta(s) = h(1 - s)\zeta(1 - s)$.

By Stirling's formula $h(s) = \exp f(s)$, where

$$f(s) = \tfrac{1}{2}(s - 1)\log \frac{s}{2\pi} - \frac{s}{2} + C_o + O(\tfrac{1}{s})$$

for $|\arg s| \le \pi - \delta$ and $|\mathrm{Im} \log(s/_{2\pi})| < \pi$.

After some calculations, the author found that the zeros of $\zeta(\frac{1}{2} + it)$ occur where on $\sigma = \frac{1}{2}$

$$\arg(h(s)\{[f'(s) + f'(1 - s)]\zeta(s) + \zeta'(s)\}) \equiv \pi/_2 \pmod{\pi} \qquad (5.7)$$

and it suffices in determining how frequently (5.7) holds to find the change in the argument of $G(s) = \zeta(s) + \zeta'(s)/[f'(s) + f'(1 - s)]$, on $\sigma = 1/_2$. Indeed, if $\arg G(\frac{1}{2} + it)$ did not change, it would follow from (5.7) and Stirling's formula that $\zeta(\frac{1}{2} + it)$ would have essentially its full quota of zeros, $N_o(T) = N(T) + O(\log T)$. In N. Levinson [5] the author showed that $\arg G(\frac{1}{2} + it)$ is sufficiently restricted so that Theorem 1 can be proved.

In Levinson [11], he gave a simplification in the proof of Levinson [5].

5.5.4 In N. Levinson [4], the author proved that the number of roots in the rectangle $R = \{\sigma + it : |\sigma - \frac{1}{2}| \le \delta , 1 \le t \le T\}$ of the equation $\zeta(s) = a$ is $= (T/_{2\pi})\log T + O_\delta(T)$, while the number of roots for which

$|\sigma - \frac{1}{2}| > \delta$, $1 \le t \le T$ is $O_\delta(T)$. The latter estimate can be improved iff $a = 0$; thus the clustering of zeros to $\sigma = 1/_2$ is more pronounced than the clustering of a-values for $a \neq 0$.

5.5.5 For a discussion of the above mentioned work of Levinson see E. Bombieri [1] in which he also gave a brief history of results concerning the zeros of the $\zeta(s)$ on the critical line.

5.5.6 Let $\mu_1(\alpha) = \limsup_{T\to\infty} \log^+ N(\alpha, T)/\log T$ and $\mu(\alpha) = \limsup_{|t|\to\infty} \log^+|\zeta(\alpha + it)|/\log |t|$. Also let $\mu(\sigma; \alpha) = \limsup'_{|t|\to\infty} \log^+(\zeta(\sigma + it)|/\log |t|$, where the prime means that t is restricted to intervals $|t - T| < \frac{1}{2}(\log T)^2$, and T is such that $\zeta(s) \neq 0$ for $\alpha > \sigma$ and $|t - T| < (\log T)^2$. The main theorem of E. Bombieri [2] stated that $\mu_1(\alpha) \le 2(1 - \alpha)[\mu(\theta)/(2\alpha - 1 - \theta) + \gamma(\alpha)]$ for $\alpha > \frac{1}{2}(\theta + 1)$, $\frac{1}{2} < \alpha \le 1$, where $\theta < 1$ is arbitrary and $-\gamma(\alpha)$ is the left derivative of $\mu(\sigma; \alpha)$ at $\alpha = \sigma$. On letting $\theta \to -\infty$, one finds that $\mu_1(\alpha) \le 2(1 - \alpha)[1 + \gamma(\alpha)]$.

5.5.7 Let $f(s) = \pi^{-s/2}\Gamma(s/_2)\zeta(s)$ and let $N_R(\lambda, T)$ and $N_I(\lambda, T)$ denote the number of zeros of the real and imaginary parts of $f(\lambda + it)$ in the interval $0 < t < T$. P.B. Brown and A. Zulauf [1] proved that for every λ satisfying $1/_2 < \lambda < 1$, there exists a positive constant A such that, for sufficiently large T , both $N_R(\lambda, T)$ and $N_I(\lambda, T)$ exceed $(2\pi)^{-1}T \log T - AT$.

5.5.8 D.R. Heath-Browne [1] proved that

(1) $N(\alpha, T) \ll T^{(4-4\alpha)/(4\alpha-1) + \varepsilon}$, $25/_{28} \le \alpha \le 1$,

(2) $N(\alpha, T) \ll T^{(3-3\alpha)/(10\alpha-7) + \varepsilon}$, $3/_4 \le \alpha \le 25/_{28}$.

5.5.9 Let $\beta_o + i\gamma_o$ be any zero of $\zeta(s)$ with $\beta_o \geq \frac{1}{2}$ and $\gamma_o \geq 100$; with every positive constant λ not exceeding $\frac{1}{2}$ and every complex number $1 + i\,\mu(\mu$ real) let $D_\lambda(1 + i\mu)$ denote the disc $|1 + i\mu - s| \leq \lambda$; then, K. Ramachandra [8] proved that there exist effective positive absolute constants C_1, C_2, C_3, C_4 (depending only on λ) such that for all y satisfying $C_1 \log\log \gamma_o \leq y \leq C_2(1 - \beta_o)^{-1}$ there holds

$$\sum_\rho e^{-y(1-\beta)} > C_3(y(1 - \beta_o))^{-1} - C_4 ,$$

where $\rho = \beta + i\gamma$ runs over all the zeros of $\zeta(s)$ which lie in $D_\lambda(1 + i\gamma_o)$ and $D_\lambda(1 + 2i\gamma_o)$.

VI. SPECIAL VALUES AT INTEGERS

§6.0. It is famous that $\zeta(2n)$, $n \geq 1$ is represented in terms of Bernoulli number and π^{2n} and so is rational up to π^{2n} . But the numerical nature of $\zeta(2n+1)$, $n \geq 1$, has long been unknown.

Let χ be a non-principal primitive character mod q and $L(s, \chi)$ a Dirichlet L-function associated with χ . Then it is known that $L(2n, \chi)$ for even χ and $L(2n+1, \chi)$, $n \geq 1$, for odd χ are represented by the generalized Bernoulli numbers. Analogously to the case of $\zeta(s)$, the numerical properties of $L(2n+1, \chi)$ for even χ and of $L(2n, \chi)$ for odd χ are unknown.

The classical formula of $\zeta(2n)$ is given in §6.5, and in §6.6, R.C.R.R. Sita and R.S.A. Siva [1], discussed six identities involving the odd integer argument of the Riemann zeta function. By using the following lemma, the author reduced the proofs of these six identities to proving just two of them:

Lemma: Let $f(x, y)$ be a real valued function defined for positive integral x and y , and let $f(x, y)$ be homogeneous of order $\alpha < -1$, i.e. for each positive integer t , $f(tx, ty) = t^{\alpha} f(x, y)$. Then the series $\sum_{r=1}^{\infty} \sum_{k=1}^{r} f(k, r)$ converges iff the series $\sum_{r=1}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} f(k, r)$ converges and in case of convergence

$$\sum_{r=1}^{\infty} \sum_{k=1}^{r} f(k, r) = \zeta(-\alpha) \sum_{r=1}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} f(k, r) .$$

Hence (6.13), (6.14), (6.15) is respectively equivalent to (6.13a), (6.14a), (6.15a). Moreover, since $\frac{1}{r^2 k} - \frac{1}{kr(k+r)} = \frac{1}{r^2(k+r)}$, we see that any two of the identities namely, (6.13) in case $a = 2$, (6.14) and (6.15). [Also (6.13a) in case $a = 2$, (6.14a) and (6.15a)] imply the third. Therefore it is sufficient to prove (6.13) and (6.14).

§6.1. Let χ denote a primitive character of modulus q , if n is a positive integer, the values $L(2n, \chi)$, when χ is even, and $L(2n-1, \chi)$, when χ is odd, are easily calculated. In particular, if $\chi(n) = (\frac{n}{p})$ where p is a odd prime, L. Carlitz [2] gave two simple formulas: for $p \equiv 3 \pmod 4$

$$L(2k+1, \chi) = \frac{(-1)^{k+1}(2\pi)^{2k+1}}{(2k+1)!\, p^{\frac{1}{2}}} \sum_{h=1}^{m} (\frac{h}{p}) B_{2k+1}(\frac{h}{p})$$

where $B_k(x)$ is the Bernoulli polynomial; for $p \equiv 1 \pmod 4$.

$$L(2k, \chi) = \frac{(-1)^{k+1}(2\pi)^{2k}}{(2k)!\, p^{\frac{1}{2}}} \sum_{h=1}^{m} (\frac{h}{p}) B_{2k}(\frac{h}{p}) \ .$$

In B.C. Berndt [4], [5], the author started from different point, but ended with same result: let $M_m(\chi) = \sum_{n=1}^{q-1} \chi(n) n^m$, if χ is even,

$$M_m(\chi) = 2G(\chi) q^m \sum_{j=1}^{[m/2]} (2\pi)^{-2j} (-1)^{j-1} \frac{m!}{(m-2j+1)!} L(2j, \bar{\chi}) \ , \qquad (6.1)$$

$$\text{where} \quad G(\chi) = G(1, \chi) = \sum_{h=1}^{q-1} \chi(h) e^{\frac{2\pi i h}{q}} \ .$$

Since $|G(\chi)|^2 = q$, if we let $m = 2$, (6.1) becomes $L(2, \chi) = (\pi^2/_{k^3}) G(\chi) M_2(\bar{\chi})$; if χ is odd,

$$M_m(\chi) = 2iG(\chi) q^m \sum_{j=1}^{[(m+1)/2]} (2\pi)^{-2j+1} (-1)^{j-1} \frac{m!}{(m-2j+2)!} L(2j-1, \bar{\chi}) \qquad (6.2)$$

if we let $m = 1$, we get

$$L(1, \chi) = (i\pi/_{q^2}) G(\chi) M_1(\bar{\chi}) \ .$$

It is clear that (6.1) and (6.2) can be used recursively to find exact formulas for $L(2n, \chi)$ and $L(2n-1, \chi)$, $n \geq 1$, respectively.

If $q \geq 3$, R. Ayoub [1] gave formulas that can calculate $L(n, \chi)$ straightly. The starting point is the power series

$$q(z, \chi) = g(z) = \sum_{n=1}^{\infty} \chi(n) z^n \quad \text{with} \quad |z| < 1 \quad \text{for convergence.}$$

If we put $n = qk + r$, then we have
$$g(z) = \sum_{r=0}^{q-1} \chi(r) z^r \sum_{k=0}^{\infty} z^{qk} = \left(\sum_{r=0}^{q-1} \chi(r) z^r\right)(1 - z^q)^{-1} .$$

We put $f(z, \chi) = f(z) = \sum_{r=0}^{q-1} \chi(r) z^r$.

Theorem 6.1. If $|z| < \frac{2\pi}{q}$, then

$$g(e^{-z}, \chi) = G(\chi) \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} (qz)^{\ell}}{(2\pi i)^{\ell+1}} \{L(\ell + 1, \overline{\chi}) + \overline{\chi}(-1)(-1)^{\ell+1} L(\ell + 1, \overline{\chi})\} .$$

Two corollaries follow at once

(i) If $\chi(-1) = 1$, then

$$g(e^{-z}, \chi) = 2G(\chi) \sum_{\substack{\ell=0 \\ \ell \text{ odd}}}^{\infty} \frac{(-1)^{\ell} (qz)^{\ell}}{(2\pi i)^{\ell+1}} L(\ell+1, \overline{\chi})$$
$$= \frac{G(\chi)}{\pi} \sum_{\ell=0}^{\infty} (-1)^{\ell} \left(\frac{qz}{2\pi}\right)^{2\ell+1} L(2\ell+2, \overline{\chi}) . \tag{6.3}$$

Likewise,

(ii) If $\chi(-1) = -1$, then

$$g(e^{-z}, \chi) = \frac{iG(\chi)}{\pi} \sum_{\ell=0}^{\infty} (-1)^{\ell+1} \left(\frac{qz}{2\pi}\right)^{2\ell} L(2\ell + 1, \overline{\chi}) . \tag{6.4}$$

We need two more simple results:

$$\sum_{m=0}^{\infty} \frac{(-1)^m M_m(\chi)}{m!} z^m = \sum_{n=1}^{q-1} \chi(n) e^{-nz} = f(e^{-z}, \chi) \tag{6.5}$$

$$\frac{1}{1-e^{-qz}} = \frac{1}{qz} + \frac{1}{2} + \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1} (qz)^{2r+1}}{(2r+2)!} \tag{6.6}$$

where B_{r+1} are the Bernoulli numbers. Combining (6.3), (6.5) and (6.6), we get in case (i) $\chi(-1) = 1$,

$$\left(\sum_{m=0}^{\infty} \frac{(-1)^m M_m(\chi)}{m!} z^m\right)\left(\frac{1}{qz} + \frac{1}{2} + \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1}(qz)^{2r+1}}{(2r+2)!}\right)$$

$$= \frac{G(\chi)}{\pi} \sum_{\ell=0}^{\infty} (-1)^{\ell} \left(\frac{qz}{2\pi}\right)^{2\ell+1} L(2\ell+2, \chi)$$

let $qz = u$ and $m = n+1$, we have

$$\left(\sum_{n=0}^{\infty} (-1)^{n+1} \frac{M_{n+1}(\chi)}{(n+1)!q^{n+1}} u^n\right)\left(1 + \frac{u}{2} + \sum_{r=0}^{\infty} \frac{(-1)^r B_{r+1} u^{2r+2}}{(2r+2)!}\right)$$

$$= \frac{G(\chi)}{\pi} \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{L(2\ell+2, \bar{\chi})}{(2\pi)^{2\ell+1}} u^{2\ell+1} \tag{6.7}$$

The expansions are valid for $|u| < 2\pi$; equating coefficients in (6.7) we infer that

$$\frac{(-1)^{\ell} G(\chi) L(2\ell+2, \bar{\chi})}{\pi(2\pi)^{2\ell+1}} = \frac{M_{2\ell+2}(\chi)}{(2\ell+2)!q^{2\ell+2}} - \frac{M_{2\ell+1}(\chi)}{2(2\ell+1)!q^{2\ell+1}}$$

$$+ \frac{M_{2\ell}(\chi)B_1}{2!(2\ell)!q^{2\ell}} - \frac{M_{2\ell-2}(\chi)B_2}{(2\ell-2)!4!q^{2\ell-2}} + \dots + \frac{(-1)^{\ell+1} M_2(\chi) B_{\ell}}{2!(2\ell)!q^2} \tag{6.8}$$

The corresponding result in case $\chi(-1) = -1$ is

$$\frac{(-1)^{\ell} iG(\chi) L(2\ell+1, \bar{\chi})}{\pi(2\pi)^{2\ell}} = \frac{M_{2\ell+1}(\chi)}{(2\ell+1)!q^{2\ell+1}} - \frac{M_{2\ell}(\chi)}{2(2\ell)!q^{2\ell}}$$

$$+ \frac{M_{2\ell-1}(\chi)B_1}{2!(2\ell-1)!q^{2\ell-1}} - \frac{M_{2\ell-3}(\chi)B_2}{(2\ell-3)!4!q^{2\ell-3}} + \dots + \frac{(-1)^{\ell+1} M_1(\chi) B_{\ell}}{(2\ell)!q} \tag{6.9}$$

§6.2. Ramanujan's formulas for L-functions.

B.C. Berndt [2] gave formulas to $L(2n, \chi)$, when χ is odd, and $L(2n - 1, \chi)$, when χ is even: Let $H = \{z : \mathrm{Im}(z) > 0\}$ and $e(z) = e^{2\pi iz}$, for $z \in H$ and s complex, define

$$A_1(z, s; \chi; r_1, r_2) = \sum_{m>-r_1} \chi(m) \sum_{n=1}^{\infty} e(n((m + r_1)z + r_2))n^{s-1},$$

$$A_2(z, s; \chi; r_1, r_2) = \sum_{m>-r_1} \sum_{n=1}^{\infty} \chi(n)e(n((m + r_1)z + r_2)/q)n^{s-1},$$

$$H_1(z, s; \chi; r_1, r_2) = A_1(z, s; \chi; r_1, r_2) + \chi(-1)e(s/_2)A_1(z, s; \chi; -r_1, -r_2);$$

$$H_2(z, s; \chi; r_1, r_2) = A_2(z, s; \chi; r_1, r_2) + \chi(-1)e(s/_2)A_2(z, s; \chi; -r_1, -r_2).$$

if $\chi(-1)(-1)^N = 1$, we have, for $j = 1, 2$,

$$H_j(z, -N; \chi; 0, 0) = 2A_j(z, -N; \chi; 0, 0) = 2A_j(z, -N; \chi).$$

Theorem 6.2 (K. Katayama [2]) Let N denote a non-negative integer and let α be an arbitrary positive number.

If $N \geq 0$ and χ is even, then

$$L(2N + 1, \chi) = \frac{2}{q}(-1)^N \alpha^{2N} G(\chi)A_1(i/_{q\alpha}, -2N; \bar{\chi}) -$$
$$- 2A_2(iq\alpha, -2N; \chi) + \frac{2}{\pi} \sum_{m=0}^{N} (-1)^{m+1} \zeta(2m)L(2N + 2 - 2m, \chi)\alpha^{2m-1}.$$

If $N \geq 1$ and χ is odd, then

$$L(2N, \chi) = -\frac{2i}{q}(-1)^N \alpha^{2N-1} G(\chi)A_1(i/_{q\alpha}, -2N + 1, \bar{\chi}) -$$
$$- 2A_2(iq\alpha, -2N + 1; \chi) + \frac{2}{\pi} \sum_{m=0}^{N} (-1)^{m-1} \zeta(2m)L(2N + 1 - 2m, \chi)\alpha^{2m-1}.$$

Theorem 6.3. Let N denote a non-negative integer and let $\rho = (-1 + i\sqrt{3})/_2$. Then if $N \geq 0$ and χ is even,

$$L(2N+1,\chi) = \frac{2}{q}(\rho/_q)^{2N} G(\chi) A_1(\rho_1 - 2N; \bar{\chi}) - 2A_2(\rho, -2N; \chi) -$$

$$- \frac{2i}{\pi} \sum_{m=0}^{N} \zeta(2m) L(2N+2-2m, \chi)\ (\rho/_q)^{2m-1} .$$

If $N \geq 1$ and χ is odd, then

$$L(2N,\chi) = \frac{2}{q}(\rho/_q)^{2N-1} G(\chi) A_1(\rho, -2N+1; \bar{\chi}) -$$

$$- 2A_2(\rho, -2N+1; \chi) + \frac{2i}{\pi} \sum_{m=0}^{N} \zeta(2m) L(2N+1-2m, \chi)(\rho/_q)^{2m-1} .$$

Both theorems 6.2, 6.3 are deduced from the transformation formulae involving $H_1(z, s; \chi; r_1, r_2)$ and $H_2(z, s; \chi; r_1, r_2)$ (B.C. Berndt [2] Theorem 3).

§6.3. If $\chi_q(q-1) = 1$, we denote the L-function $L(s, \chi_q)$ by $L_{+q}(s)$, if $\chi_q(q-1) = -1$, we denote it $L_{-q}(s)$. Then, the functional equations for primitive L-functions become:

$$L_{-q}(s) = C(s)\cos(s\pi/_2) L_{-q}(1-s)$$

$$L_{+q}(s) = C(s)\sin(s\pi/_2) L_{+q}(1-s)$$

$$\text{where } C(s) = 2^s \pi^{s-1} q^{-s+\frac{1}{2}} \Gamma(1-s) .$$

The above equations allow us to calculate $L_{\pm q}$ for all real s. It is well known that $L_{+1}(-2m) = 0$ and $L_{+1}(2m) = R\pi^{2m}$ where m is a positive integer and R a rational number. Similarly, it is shown by M.L. Glasser [1] that $L_{-4}(1-2m) = 0$ and $L_{-4}(2m-1) = R'\pi^{2m-1}$ with R' rational. In I.J. Zucker and M.M. Roberson [1], the authors gave some special cases of the following result:

Theorem 6.4. For m a positive integer

(a) $L_{-q}(1-2m) = 0$, $L_{-q}(2m-1) = R'q^{-\frac{1}{2}}\pi^{2m-1}$,

$L_{-q}(-2m) = (-1)^m R'(2m)!/(2q)^{2m}$;

(b) $L_{+q}(-2m) = 0$, $L_{+q}(2m) = Rq^{-\frac{1}{2}}\pi^{2m}$,

$L_{+q}(1-2m) = (-1)^m R(2m-1)!/(2q)^{2m-1}$,

where R and R' are rational numbers depending on m and q .

$L_{-k}(1-2m) = L_{+k}(-2m) = 0$ follows immediately from functional equations and the fact that $\Gamma(1-s)$ has simple poles whenever s is a positive integer.

By applying Fubini's theorem to $L_{\pm q}(s)\Gamma(s)$ and a change of contour one gets

$$L_{\pm q}(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-w)^{s-1}}{1-e^{-qw}}\left(\sum_{n=1}^{q}\chi_q(n)e^{-nw}\right)dw .$$

The rest of the results now follow the residue theorem.

Nothing general appears to be known about $L_{-q}(2m)$ and $L_{+q}(2m-1)$. For example, it is not known how to express $L_{+1}(3) = \zeta(3)$ or $L_{-4}(2)$ in terms of known transcendentals. Up to now each $L_{-q}(2m)$ and $L_{+q}(2m-1)$ has been considered a new constant, with $L_{-4}(2)$ having the status of being named Catalan's constant. However, it is possible to express all $L_{\pm q}(1)$ in terms of known transcendentals. It had been shown in the above theorem that $L_{-q}(1) = R'q^{-\frac{1}{2}}\pi$. This is just one part of a remarkable result of Dirichlet's concerning the class number $h(d)$ of the binary quadratic form $am^2 + bmn + cn^2$ with discriminant $d = b^2 - 4ac$. Dirichlet showed that if

$$d < 0 \quad L_{-d}(1) = h(d)\pi/d^{\frac{1}{2}} \tag{6.10}$$

$$d > 0 \quad L_{+d}(1) = 2h(d)\ln \varepsilon_0/d^{\frac{1}{2}} \tag{6.11}$$

(6.10) tells us that the R' in $L_{-q}(1) = R'q^{-\frac{1}{2}}\pi$ is a whole number since $h(d)$ is a whole number. In (6.11), ε_o is the fundamental unit in the quadratic number-field $Q(\sqrt{d})$. An account of this may be found in H.M. Stark [1]. ε_o is easily found for any given real quadratic field and thus $L_{+q}(1)$ may be expressed in known transcendentals. For example

$$L_{+5}(1) = 1 - 2^{-1} - 3^{-1} + 4^{-1} + 6^{-1} - 7^{-1} - 8^{-1} + 9^{-1} \dots = \frac{1}{\sqrt{5}} \ln\left(\frac{3+\sqrt{5}}{2}\right)$$

$$L_{+13}(1) = 1 - 2^{-1} + 3^{-1} + 4^{-1} - 5^{-1} - 6^{-1} - 7^{-1} - 8^{-1} + 9^{-1} + 10^{-1} - 11^{-1} + 12^{-1} \dots$$

$$= \frac{1}{\sqrt{13}} \ln\left(\frac{11+3\sqrt{13}}{2}\right) .$$

§6.4. In E. Grosswald [5], explicit representations are found for $L(a, \overline{\chi})\zeta(a)$. We introduce some notations first: let $\chi(n)$ be a primitive character to the modulus q and $a = 2m+1$ is an odd natural integer. Set $\phi(s) = \zeta(s)\zeta(s+a)L(s, \chi)L(s+a, \overline{\chi})$ and $\Phi_o(s) = (\frac{4\pi^2}{q})^{-s}\phi(s)\Gamma^2(s)$. If $\chi(-1) = 1$, $\Phi_o(s)$ has double poles at $s = -1, -3, \dots, -a+2$. Let $\Phi_1(s) = p_1(s)\Phi_o(s)$ with $p_1(s) = (s+1)(s+3)\dots(s+a-2)$. Then, $\Phi_1(s)$ has only simple poles at $s = 1, 0, -1, -2, \dots, -a$. For $c > 1$, define

$$F_1(\tau) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \Phi_j(s)(\tau/_i)^{-s} ds .$$

Finally, let $\beta_o = (a-1)F_1(i) - 2iF_1'(i)$ and $\beta_1 = 2F_1(i)$. Now we state the results as follows:

Theorem 6.5. If $\chi(-1) = 1$ and $a \equiv 1 \pmod 4$, then

$$\pi^{-2a}L(a, \overline{\chi})\zeta(a) = \frac{\tau(\overline{\chi})L(1, \chi)}{\tau(\chi)L(1, \overline{\chi})} V_1(a) + \frac{\tau(\overline{\chi})}{\tau(\chi)L(1, \overline{\chi})} V_2(a)$$

$$+ \frac{\beta_o}{\pi^{-2a}\tau(\chi)L(1, \overline{\chi})} V_3(a)$$

with algebraic $V_j(a)$ $(j = 1, 2, 3)$.

Corollary. If under the conditions of Theorem 1, $\chi(n)$ is a real character; then $\chi(n) = \overline{\chi}(n)$ and

$$\pi^{-2a}L(a, \chi)\zeta(a) = R_1(a) + \frac{R_2(a)}{L(1, \chi)} + \frac{\pi^{-2a}\beta_o}{\tau(\chi)L(1, \chi)} R_3(a)$$

with rational $R_j(a)$ $(j = 1, 2, 3)$.

Theorem 6.6. If $\chi(-1) = 1$ and $a \equiv 3 \pmod 4$, then

$$\pi^{-2a}L(a, \overline{\chi})\zeta(a) = \frac{\tau(\overline{\chi})L(1, \chi)}{\tau(\chi)L(1, \overline{\chi})} v_1(a) + \frac{\tau(\overline{\chi})}{\tau(\chi)L(1, \overline{\chi})} v_2(a) + \frac{\beta_1}{\pi^{2a}\tau(\chi)L(1, \overline{\chi})} v_3(a)$$

with algebraic $v_j(a)$ $(j = 1, 2, 3)$.

Corollary. If under the conditions of Theorem 2, $\chi(n)$ is a real character, then $\chi(n) = \overline{\chi}(n)$ and

$$\pi^{-2a}L(a, \chi)\zeta(a) = r_1(a) + \frac{r_2(a)}{L(1, \chi)} + \frac{\beta_1 r_3(a)\pi^{-2a}}{\tau(\chi)L(1, \chi)}$$

with rational $r_j(a)$ $(j = 1, 2, 3)$.

If $\chi(-1) = -1$, a similar statement holds, however, as the unknown of arithmetic nature, the author does not pursue the matter further.

§6.5. The classical formula

$$\zeta(2n) = \sum_{k=1}^{\infty} k^{-2n} = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!} \quad (n \geq 1) \tag{6.12}$$

is one of the most beautiful results of elementary analysis. The numbers B_n are Bernoulli numbers and can be defined by the recursion formula

$B_o = 1$, $B_n = \sum_{s=0}^{n} \binom{n}{s} B_s$ for $n \geq 2$.

Several writers have given more elementary proofs of (6.12) that do not require concepts from advanced real or complex analysis. Recently, T.M. Apostol [1] and B.C. Berndt [1] gave some new elementary proofs of (6.12).

In the former paper, the author extended the method of evaluating $\zeta(s)$ of P. Ioannis [1] to the evaluation of $\zeta(2n)$. The key ingredient in P. Ioannis [1] is the formula:

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) = \frac{m(2m-1)}{3}$$

or rather the asymptotic relation

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) = \frac{2}{3} m^2 + O(m) \tag{6.13}$$

which it implies. The author made use of the following lemma which provides a generalization of (6.13).

Lemma: For any integer $m \geq 1$, $n \geq 1$, we have

$$\sum_{k=1}^{m} \cot^{2n}\left(\frac{k\pi}{2m+1}\right) = (-1)^{n-1} \frac{2^{4n-1} B_{2n}}{(2n)!} m^{2n} + O(m^{2n-1}) , \tag{6.14}$$

where the constant implied by the O-symbol is independent of m .

Once the lemma is proved, the evaluation is routed.

In B.C. Berndt [1], the author described two elementary proofs of (6.12). The first proof is based on the equation:

$$\lim_{N\to\infty} \int_0^a f(x)\frac{\sin(NX)}{\sin X}\, dx = 0$$

where $f(x)$ is twice continuously differentiable on $[0, a]$, $0 < a < \pi$, and suppose that $f(0) = 0$.

The second proof, similar as T.M. Apostol [1], starting from an trigonometrical equation:

$$\pi^2 \csc^2(\pi x) = \sum_{k=-\infty}^{\infty} (k + x)^{-2} .$$

As it can be seen, the properties of Bernoulli numbers play an important role among the proofs of both papers.

Moreover, P.M. Chen [1] proved that for each positive n, $\zeta(2n) = \sum_{k=1}^{\infty} k^{-2n} = An(\pi/_2)^{2n}$ where $A_1 = 2/_3$ and $An = \sum_{r=1}^{n-1} (-1)^{r+1} 2^{2r} A_{n-r}/_{(2r+1)!} + (-1)^{n+1} 2^n \cdot n/_{(2n+1)!}$ for $n > 1$.

For earlier results on $\zeta(2n)$ see Estermann [1], Kuo [1], Carlitz [1].

§6.6. From E. Grosswald [4], it is known that for $a = 2n + 1$, $\zeta(a) = \pi^a r(a) - G_a(i)$, where $G_a(\tau)$ has a Fourier expansion $G_a(\tau) = \sum_{n=1}^{\infty} c_n(a) e^{2\pi i n \tau}$ and $r(a)$ is rational. In E. Grosswald [1], the author showed that for a slightly more general function $G_a(\tau, \chi) = \sum_{n=1}^{\infty} c_n(a, \chi) e^{2\pi i n \tau/k}$ [which, for $k = 1$ reduces to $G_a(\tau)$], if $\chi(n)$ is a nonprincipal; primitive congruence character modulo k, then $G_a(i, \chi) = \pi^a \tau(\bar{\chi}) r(a, \chi)$. Here, $\bar{\chi}$ is the conjugate character, $\tau(\bar{\chi})$ is a Gaussian sum, and $r(a, \chi)$ is algebraic. If $\chi(n)$ is also real, then $k^{-\frac{1}{2}} \pi^{-a} G_a(i, \chi)$ is actually rational.

As in §6.5, let B_n be the Bernoulli numbers, we consider the following formula $R(\nu, \rho)$ with (i), (ii) below, for every positive integer $\nu \geq 1$ and for a positive real ρ :

$$R(\nu,\ \eta)\ :\ \frac{1}{(4\alpha)^{\nu}}\{\frac{1}{2}\zeta(2\nu+1)+\sum_{m=1}^{\infty}\frac{1}{m^{2\nu+1}(e^{2\alpha m}-1)}\}\ -$$

$$-\frac{1}{(-4\beta)^{\nu}}\{\frac{1}{2}\zeta(2\nu+1)+\sum_{m=1}^{\infty}\frac{1}{m^{2\nu+1}(e^{2\beta m}-1)}\}$$

$$=\frac{B_{2\nu+2}}{(2\nu+2)!}\{(-\alpha)^{\nu+1}+\beta^{\nu+1}\}\ -$$

$$-\sum_{k=1}^{[\frac{1}{2}(\nu+1)]}(-1)^{k}\pi^{2k}\frac{B_{2k}}{(2k)!}\frac{B_{2\nu+2-2k}}{(2\nu+2-2k)!}\{(-\alpha)^{\nu+1-2k}+\beta^{\nu+1-2k}\}\ .$$

(i) $\alpha = \pi/\eta$, $\beta = \pi\eta$, $\eta > 0$

(ii) if ν is odd, the term corresponding to $k = \frac{1}{2}(\nu + 1)$ in the last summation is multiplied by $\frac{1}{2}$.

In E. Grosswald [4], the author proved

(G1) $R(\nu,\ 1)$ is valid for positive odd integer ν .

(G2) the formula obtained by differentiating $R(\nu,\ \eta)$ with respect to η is valid for positive even integers ν and $\eta = 1$.

and the purpose of K. Katayama [1] is to prove, as an application of the theory of the zeta-theta function, that $R(1,\ \eta)$ and $R(2,\ \eta)$ are valid. Here, the zeta-theta function $\zeta_j(w,\ s)$ for $w = \xi + i\eta$ $\eta > 0$, $s \in C$, is defined as follows:

$$\zeta_j(w,\ s) = \theta_j(w)z(w,\ s)\ ,\quad j = 0,\ 1\ .$$

$$\theta_0(w) = \Sigma e^{-2\pi i\bar{w}m^2}\ ,\quad \theta_1(w) = \Sigma e^{-2\pi i\bar{w}(m+\frac{1}{2})^2}\ ,$$

$$z(w,\ s) = \frac{\Gamma(s/2)}{\pi^{s/2}\eta^{s/2}}\zeta(s) + \frac{\Gamma((s+1)/2)}{\pi^{(s+1)/2}}\eta^{s/2}\zeta(s+1)$$

$$+\sum_{\substack{m\neq 0\\ n\neq 0}} e^{2\pi i\xi mn}\left|\frac{n}{m}\right|^{s/2}K_{s/2}(2\pi\eta|mn|)\ ,$$

$$(\text{for } \operatorname{Re} s > 2)\ .$$

where $K_{\mu}(z)$ is the so-called modified Bessel function.

The proof goes on the following way, for $\nu = 1$: Differentiate $R(1, \eta)$, to be proved, with respect to η . Then it coincides with the inversion formula of the zeta-theta function at $s = 3$. Therefore the integral of the latter with respect to η is equal to $R(1, \eta)$ up to some constant. To determine it, the author used (G1) .

For $\nu = 2$, the proof will go on the same line.

In E. Grosswald [3], the author proved a more general theorem:

For real positive α, β with $\alpha\beta = \pi^2$ and $s = \sigma + it$ consider the function

$$F(s; \alpha, \beta) = (-\alpha)^{1-s}\{\tfrac{1}{2}\zeta(2s-1) + \sum_{m=1}^{\infty} m^{1-2s}(e^{2\alpha m} - 1)^{-1}\} -$$
$$- \beta^{1-s}\{\tfrac{1}{2}\zeta(2s-1) + \sum_{m=1}^{\infty} m^{1-2s}(e^{2\beta m} - 1)^{-1}\} -$$
$$- \frac{1}{2}\sum_{\nu=0}^{[\sigma]}(-1)^{\nu}\zeta(2\nu)\zeta(2s-2\nu)\pi^{2\nu-2s}\{(-\alpha)^{s-2\nu} + \beta^{s-2\nu}\} ;$$

where the last sum is equal to zero for $\sigma < 0$, and where $(-\alpha)^{1-s}$ is defined by $\{\exp (1-s) \log(-\alpha)\}$ with some definite (but arbitrary) determination of the logarithm: Then $F(n; \alpha, \beta) = 0$ for all rational integers $n \neq 1$.

For the case $n = 1$, the author gave another theorem: For $\alpha\beta = \pi^2$ and real s ,

$$\lim_{s\to 1^+} F(s; \alpha, \beta) = \frac{1}{4}\log(-\beta/_{\alpha}) + (\beta - \alpha)/_{12} + \frac{1}{2}\sum_{m=1}^{\infty} m^{-1}(\coth \alpha m - \coth \beta m) ,$$
$$\lim_{s\to 1^-} F(s; \alpha, \beta) = \frac{1}{4}\log(-\beta/_{\alpha}) + (\beta - \alpha)/_{24} + \frac{1}{2}\sum_{m=1}^{\infty} m^{-1}(\coth \alpha m - \coth \beta m) .$$

For $\alpha = \beta = \pi$, in particular, $\lim_{s\to 1} F(s; \pi, \pi) = \frac{1}{4}\log(-1) \neq 0$ the determination of $\log(-1)$ being consistent with the former theorem.

In R.C.R.R. Sita and R.S.A. Siva [1] the author discussed six identities involving the Riemann's zeta function:

$$2\sum_{r=1}^{\infty}\frac{1}{r^a}\sum_{k=1}^{r}\frac{1}{k} = (a+2)\zeta(a+1) - \sum_{i=1}^{a-2}\zeta(a-i)\zeta(i+1) \tag{6.13}$$

$$2\sum_{r=1}^{\infty}\frac{1}{r^a}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{k} = a+2-\frac{1}{\zeta(a+1)}\sum_{i=1}^{a-2}\zeta(a-i)\zeta(i+1) \tag{6.13a}$$

$$\sum_{r=1}^{\infty}\sum_{k=1}^{r}\frac{1}{kr(k+r)} = \frac{5}{4}\zeta(3) \tag{6.14}$$

$$\sum_{r=1}^{\infty}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{kr(k+r)} = \frac{5}{4} \tag{6.14a}$$

$$\sum_{r=1}^{\infty}\sum_{k=1}^{r}\frac{1}{r^2(r+k)} = \frac{3}{4}\zeta(3) \tag{6.15}$$

$$\sum_{r=1}^{\infty}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{r^2(r+k)} = \frac{3}{4}\ . \tag{6.15a}$$

The identity (6.13) is originally due to Williams (cf. M.S. Klamkin [2] p.130) and in case $a = 2$ reduces to a result doe to S. Chowla [6]. The special case $a = 3$ appears as a problem proposed by M.S. Klamkin [1] (6.13a) in case $a = 2$ reduces to a result due to R.J. Hans and V.C. Dumir [1], while (6.15a) is due to H. Gupta [1].

The author reduced the proofs of these six identities to proving just two of them (see §6.0). A proof of these two identities, based on a generalization of a transformation formula due to J. Lehner and M. Newman [1] is given as follows:

Lemma (A transformation formula): Let $f(x, y)$ be a real valued function defined for positive integral x and y. For a fixed positive integer n let T_n be the set of all ordered pairs (r, k) of positive integers that $1 \le r, k \le n$; $r + k \ge n + 1$ and $S_n = \sum_{(x,y)\in T_n} f(x, y)$. Then

$$S_n = \sum_{r=1}^{n} f(r, r) + \sum_{r=2}^{n} \sum_{k=1}^{r-1} \{f(k, r) + f(r, k) - f(k, r-k)\} \qquad (6.16)$$

To prove (6.13) and (6.14), the author took $f(x, y) = \frac{1}{x^a y}$ and $f(x, y) = \frac{2y + x}{2x^2 y(x+y)}$ in (6.16) respectively.

Similar as the evaluation of $\zeta(2n)$, T.M. Apostol [1] showed that

$$\zeta(2n + 1) = \left(\frac{\pi}{2}\right)^{2n+1} \lim_{m\to\infty} \frac{1}{m^{2n+1}} \sum_{k=1}^{m} \cot^{2n+1}\left(\frac{k\pi}{2m+1}\right) .$$

Starting from formulas like

$$B_n(x)/_{n!} = -\sum_{\nu=-\infty}^{\infty} \exp(2\pi i\nu x)/(2\pi i\nu)^n \quad (n \ge 2,\ 0 \le x \le 1)$$

concerning the Fourier series expansion of Bernoulli polynomials, H. Schmidt [1] gave simple proofs for several integral representations of arithmetic values, such as
$\zeta(2k + 1) = ((-1)^{k-1}(2\pi)^{2k+1}/(2k + 1))! \int_0^{\frac{1}{2}} B_{2k+1}(x) \cot \pi x dx \ (k \ge 1)$, of the Riemann zeta function. Some applications of these results are also discussed.

In A. Terras [1], the author derived some formulas for the Riemann zeta-function $\zeta(2N+1)$, with particular emphasis on the cases $N = 1$ and α .

VII. OTHER ZETA FUNCTIONS.

§7.1. Hurwitz zeta-functions

The Hurwitz zeta function is defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} (a + n)^{-s} \quad 0 < a \le 1, \quad \sigma > 1, \ s = \sigma + it . \qquad (7.1)$$

7.1.1 R. Spira [3] calculated the zeros of $\zeta(s, a)$ for $a = \frac{1}{3}, \frac{2}{3}$, $|t| \le 100$, $\sigma \ge -5$. Moreover, the author proved that

(1) If $\sigma \ge 1 + a$, then $\zeta(s, a) \ne 0$.

(2) If $\sigma \le -1$ and $|t| \ge 1$, then $\zeta(s, a) \ne 0$.

(3) If $\sigma \le -(4a + 1 + 2[1 - 2a])$ and $|t| \le 1$, then $\zeta(s, a) \ne 0$ except for trivial zeros on the negative real axis, one in each interval

$$-2n - 4a \pm 1 , \quad n \ge 1 - 2a .$$

7.1.2 In I.M. Kalnin [1], the author obtained a simple derivation of the Hurwitz formula for the function $\zeta(s, a)$ by using the Euler-Maclaurin summation formula to evaluate the sum in the relation

$$\Gamma(s)\zeta(s, a) = \int_0^{\infty} t^{s-1} \sum_{n=0}^{\infty} e^{-(n+a)t} dt .$$

7.1.3 In R. Balasubramanian [1], the author showed that

$$\int_0^1 |\zeta_1(\tfrac{1}{2} + it, \alpha)|^2 d\alpha = \log t + O(\log\log t) . \quad \text{(cf: Koksma \& Lekkerkerker [1])}$$

7.1.4 Let $\theta(z, v, a, b) = \sum_{n=0}^{\infty} (n + a) z^{(n+a)b}$, where $|z| < 1$, $0 < a \le 1$, $b > 0$ and v is arbitrary. B.R. Johnson [1] used Mellin transforms to obtain a contour integral representation for θ and then revaluates the integral by residue calculus to obtain the formula

$$\theta(z, v, a, b) = b^{-1}\Gamma((1-v)/b)\beta^{(v-1)/b} + \sum_{r=0}^{\infty}((-1)^r/r!)\rho(v-br, a)\beta^r ,$$

where $\zeta(s, a)$ is the Hurwitz zeta function, $\beta = \log 1/_z$, $b \leq 1$ and $v - br \neq 1$ for $r = 0, 1, 2, \ldots$. If $b = 1$ and the added requirement that $|\log z| < 2\pi$, and the formula reduces to a known result for the Lerch zeta function.

7.1.5 J. Hadamard [Bull Soc. Math. France, 37(1909), 59-60] proved that each of the two functional equations:

$$(2^{s-1} - 1)f(s) = \sum_{n=1}^{\infty}(\Gamma(s)n/(n!\ 2^{n+1}))f(s + n) ,$$

$$(2^{s-1} - 1)f(s) = 2^{s-1} + {}_{n=1}(-1)^n(\Gamma(s)n/(n!\ 2^{n+1}))f(s + n)$$

with the additional condition $\lim_{n\to\infty} f(s + n) = 1$ has as unique solution the Riemann zeta function $\zeta(s)$. A. Dinghas [1] gave a new proof of this result and proved an analogous characterization of the Hurwitz zeta function.

7.1.6 In E.E. Brenev [1], the author proved that for $\operatorname{Re} w > 0$ and nonnegative integral λ one has

$$\zeta'(-\lambda, w) = \log wB_\lambda(w)/(1 + \lambda) - P_\lambda(w)$$
$$+ \sum_{\alpha=2}^{\infty}\zeta(\alpha, 1 + w)\int_0^1 x^{\alpha-1}B_\lambda(x)/(1 + \lambda)dx$$

where P_λ is a polynomial of degree $1 + \lambda$ and B_λ is the λ-th Bernoulli polynomial. In E.E. Breneu [2] the author transformed that identity and obtained various formulas for $\zeta'(-\lambda, w)$.

§7.2. Lerch zeta-function

The Lerch zeta function $\zeta(s; x, w)$ is defined by the series

$$\zeta(s; x, w) = \sum_{n=1}^{\infty}e(nw)(x + n)^{-s} , \quad e(t) = e^{2\pi it} , \tag{7.2}$$

for $\mathrm{Re}\,(s) > 0$, where x and w are real numbers, w is not an integer, x is not a non-positive integer (see Lerch [1]).

7.2.1 Benneton [1] considered the function defined by

$\zeta_a(s, u) = \sum_{n=1}^{\infty} e^{-2\pi i n u}(n - a)^{-s}$, $0 \le a < 1,\ 0 \le u < 1,\ \sigma > 1,\ s = \sigma + it$. The author obtained its analytic continuation in the whole s-plane, dervies a functional equation, evaluates the function when s is a negative integer, and further examines the special case with $u = 0$. The author was apparently unaware that, with a change of notation, $\zeta_a(s, u)$ is the Lerch zeta-function.

7.2.2 M. Mikolas [1] presented a short proof of the functional equation of the Lerch zeta-function that is based on the Fourier expansion of $\zeta(1 - s;\ x,\ w)e(xw)$, $0 < \mathrm{Re}\ (5) < 1$, as a function of x over $(0, 1)$. (For other proofs see Apostol [9], Oberhettinger [1]). From this functional equation the author deduced functional equations for other zeta-functions.

§7.3. Secondary zeta functions

The secondary zeta-functions $Z_p(s)$ and $Z_\gamma(s)$, whose properties we are going to investigate, are defined by:

$$Z_p(s) = \lim_{T\to\infty}\Big\{\sum_{0<m\log p<T} \frac{\log p}{p^{\frac12 m}}(m \log p)^{-s} - \int_0^T e^{\frac12 u} u^{-s}\, du\Big\} ,$$

where p runs through the prime numbers and m through the positive integers, and

$$Z_\gamma(s) = \sum_{\mathrm{Re}(\gamma)>0} \gamma^{-s} .$$

7.3.1 I.C. Chakravarty [1] showed that the secondary zeta function $Z_\gamma(s)$ has a double pole at $s = 1$ with principal part

$$(1/2\pi(s - 1)^2) - (\log 2\pi)/(2\pi(s - 1)) ,$$

$Z_\gamma(-2m) = (-1)^m(8 - E_{2m})/2^{2m+3}$, m being a non-negative integer, and that $Z_p(s)$ has simple poles with residues $1/(2^{m-1}(m)!)$ at $s = m$, m being a positive integer, $Z_p(0) = 2 - (\zeta'(\frac{1}{2})/\zeta)$, $Z_p(-2m) = (2m)!(2^{2m+1} - (2^m - \frac{1}{2})\zeta(2m + 1)) - \frac{1}{4}(-1)^m \pi^{2m+1} E_{2m}$. Here E_{2m} denotes an Euler number in even subscript notation.

7.3.2 In I.C. Chakravarty [2], define

$$Z_\gamma(s) = \lim_{T\to\infty}\{\Sigma_{0<\gamma<T}\gamma^{-s} - \frac{T^{1-s}}{2\pi(1-s)}\log\frac{T}{2\pi} + \frac{T^{1-s}}{2\pi(1-s)^2}\}$$

$(\mathrm{Re}(s) > 0)$ and

$$\zeta_\gamma(s) = \lim_{h\to+0}\{\Sigma_{\mathrm{Re}(\gamma)\to 0}\gamma^{-s}e^{-\frac{1}{2}\gamma^2 h} - (I(s, h)/j(2\pi)2^{\frac{1}{2}s}\Gamma(\frac{1}{2}s))\}$$

where $I(s, h) = \Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2} - \frac{1}{2}s)h^{\frac{1}{2}s-1}\{\frac{\Gamma'(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2} - \frac{1}{2}s)} - \log 2\pi^2 h\}/4\sqrt{\pi}$.

Similarly, define

$$Z_p(s) = \lim_{T\to\infty}\{\Sigma_{\log n<T}(\Lambda(n)/n^{\frac{1}{2}})(\log n)^{-s} - \int_0^T x^{-s}e^{\frac{1}{2}x}dx\}$$

$(\mathrm{Re}(s) > \alpha)$, and

$$\zeta_p(s) = \lim_{h\to+0}\{\Sigma_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\frac{1}{2}}}(\log n)^{-s}e^{-\frac{1}{2}h(\log n)^2} - \frac{\sqrt{(2\pi)}f(s, h)}{2^{\frac{1}{2}s}\Gamma(\frac{1}{2}s)}\}$$

where $f(s, h) = \int_0^\infty(t^{\frac{1}{2}s-1}e^{1/8(t+h)}/\sqrt{(t+n)})dt$.

The author observed that under the Riemann hypothesis $Z_\gamma(s) = \zeta_\gamma(s)$ and $Z_p(s) = 2^{1-s}\Gamma(1-s)\zeta_p(s)$, thus explaining the difference between the form of the functional equation connecting Z_γ and Z_p and that connecting ζ_γ and ζ_p .

7.3.3 In N. Levinson [13], the author obtained the following results without any hypothesis: Let

$Z(s, b) = \Sigma_2^{\infty}(\Lambda(n) - 1)_n^{-\frac{1}{2}}(\log n)^{-s}\exp\{-b \log n \log \log ne\}$ $(b > 0)$; $\rho = \beta + i\gamma$ the zeros of $\zeta(s)$ in $|\sigma| < 1$, $q = i(\frac{1}{2} - \rho) = \gamma + i(\frac{1}{2} - \beta)$; $Y(s) = \Sigma_{\mathrm{Re}\ q>0}\, q^{-s}$; $L_1(s) = \Sigma_o^{\infty}(2n + 1)^{-s}$, $L_2(s) = \Sigma_o^{\infty}(-1)^n(2n + 1)^{-s}$; $H(s) = \int_o^{\infty}(e^y - [e^y])e^{-y/2}y^{-s}dy$ $(\sigma < 2)$; then $Z(s) = \lim_{b\to o}Z(s, b)$ as an entire function; $Y(s)$ is meromorphic, the singularities being a double pole at $s = 1$ and simple poles at $s = -2n - 1$ $(n \geq 0)$; $H(s)$ is meromorphic, the singularities being simple poles at $s = 2n$ $(n \geq 1)$; in the finite s-plane the following equation holds:

$$\cos \pi s/2\Gamma(s)Z(s) = -\pi Y(1-s) - \pi 2^{-s-1}\{L_1(1-s) + L_2(1-s) - 2\}/(5m\ \pi s/2)$$
$$+ \cos \pi s/2\ \Gamma(s)\{SH(s+1) + \tfrac{1}{2}H(s)\}$$

or (using $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$) $= -2 \sin \pi s/2\ \Gamma(1-s)Y(1-s) - \Gamma(1-s)2^{-s}\{L_1(1-s) + L_2(1-s) - 2\} + sH(s+1) + \frac{1}{2}H(s)$. Results concerning the asymptotic nature of $Z(s)$ are also given.

Let $F(s, \chi) = \Sigma_{N+1}^{2N}a(m)\chi(m)m^{-s}$,

$$E = \Sigma_{q\leq Q,\ q_o|q}\ \Sigma^*_{\chi \bmod q}\ \Sigma_{r=1}^{R(x)}\ |F(s(r, \chi), \chi)| ,$$

where $s(r, \chi) = \sigma(r, \chi) + it(r, \chi)$, $0 \leq \sigma(r, \chi) \leq \log^{-1}(NQ^2T)$, $1 \leq |t(r_1, \chi_1) - t(r_2, \chi_2)| \leq T$ (the lower bound applying only for $\chi_1 = \chi_2$ and $r_1 \neq r_2$) , $D = Q^2T/_{q_o}$, $G = \Sigma_{N+1}^{2N}|a(m)|^2$, $R = \Sigma_{q\leq Q,\ q_o|q}\Sigma^*_{\chi \bmod q}R(\chi)$. For each triple R, M, D of real numbers ≥ 1 , let $B(R, M, D)$ be the least positive number such that $E \leq G^{\frac{1}{2}}B(R, M, D)$ for every choice of q_o, Q, T of $N \leq M$, of coefficient $a(m)$ and sets of points $s(r, \chi)$ satisfying the above relations. Then M.N. Huxley [1] showed that $B(R, M, D)$ is at most $R^{\frac{1}{2}}N^{\frac{1}{2}} + RN^{\frac{1}{4}} + R^{1-\frac{1}{2}k}N^{\frac{1}{4}}B(R, D^k/N^k, D)^{\frac{1}{2}k}$, where k is a positive integer.

As an application the author proved various zero-density estimates for L-functions, for example, the "Q^2T" density hypothesis in $\sigma \geq 5/_6$ and the density hypothesis for $\zeta(s)$ in $\sigma \geq 189/_{230}$.

§7.4. Epstein zeta function

Let $s = \sigma + it$ ... Then in $\sigma > p$, the Epstein zeta function of the p-th order with "upper" parameters $g_1, \ldots, g_p$ and "lower" parameters $h_1, \ldots, h_p$ is defined by

$$Z\begin{vmatrix} g_1, \ldots, g_p \\ h_1, \ldots, h_p \end{vmatrix}(s) = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_p=-\infty}^{\infty} \frac{\exp(2\pi i \sum_{j=1}^{p} k_j h_j)}{(\sum_{1\leq j\leq p} (k_j+g_j)^2)^{s/2}} . \qquad (7.3)$$

This function may then be defined in the whole plane by analytic continuation: it is entire unless all h_j are integers, in which case there is a simple pole $s = p$.

If all the $g_j = g$ and all the $h_j = h$, denote the left side of (7.3) by $Z\begin{vmatrix} g \\ h \end{vmatrix}(s)$. In O. Emerskeben [1], the author investigated the function ${}^{(p)}f(s)$ appearing in the functional equation

$${}^{(p)}Z\begin{vmatrix} o \\ 1/_2 \end{vmatrix}(s) = {}^{(p)}f(s)\,{}^{(p)}Z\begin{vmatrix} o \\ o \end{vmatrix}(s) .$$

He first defined the entire function ${}^{(p)}f^*(s)$ appearing in the equation ${}^{(p)}Z\begin{vmatrix} 1/2 \\ o \end{vmatrix}(s) = {}^{(p)}f^*(s)\,{}^{(p)}Z\begin{vmatrix} o \\ o \end{vmatrix}(s)$, and showed that ${}^{(p)}f^*(s) = {}^{(p)}f(p-s)$. In O. Emersleben [2], he defined ${}^{(p)}g(s)$ by ${}^{(p)}Z\begin{vmatrix} 1/2 \\ o \end{vmatrix}(s) - {}^{(p)}Z\begin{vmatrix} o \\ o \end{vmatrix}(s) = {}^{(p)}g(s)\,{}^{(p)}Z\begin{vmatrix} o \\ 1/_2 \end{vmatrix}(s)$, he gave explicit closed expressions for ${}^{(1)}g$, ${}^{(2)}g$, ${}^{(4)}g$ and ${}^{(8)}g$. The author also discussed applications to representations by sums of squares using the Dirichlet series representation of these functions: clearly

$$ {}^{(p)}Z\begin{vmatrix} 0 \\ 0 \end{vmatrix}(s) = \sum_{n=1}^{\infty}(r_p(n))/n^{s/2} \quad \text{and} \quad {}^{(p)}Z\begin{vmatrix} 0 \\ 1/2 \end{vmatrix}(s) = \sum_{n=1}^{\infty}(-1)^n r_p(n)/n^{s/2} , $$

and the author finds similar but more complicated expressions for

${}^{(p)}Z\begin{vmatrix} 1/2 \\ 0 \end{vmatrix}(s)$, $p = 2, 4, 8$; here, as usual, $r_p(n)$ denotes the bumber of representations of n as the sum of p squares.

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